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Practical aspects of modelling parameter uncertainty for risk capital calculation

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Abstract We assume that an insurance undertaking models its risk by a random variable $\boldsymbol{X} = \boldsymbol{X}(\theta)$ with a fixed parameter (vector) θ. If the undertaking does not know θ, it faces parameter uncertainty (see e.g. [1,2,4,5,10]). It is well-known that neglecting parameter uncertainty can lead to an underestimation of the true risk capital requirement.

In this contribution we address some practical questions. A risk capital requirement not taking into account parameter uncertainty can imply a probability of solvency significantly below the required confidence level. However, the underestimation of the confidence level depends on the distribution, the size of the sample and, in general, on the true parameters of the distribution. We determine the probability of solvency for different distributions and samples sizes.
We then follow the "inversion method" introduced in [4], which is known to model an appropriate risk capital requirement respecting parameter uncertainty for a wide class of distributions and common estimation methods. We extend the idea to distribution families and estimation methods that have not been considered so far but are frequently used to model the losses of an insurance undertaking: the lognormal distribution together with the method of moments and the two-parameter gamma distribution. Experimental data demonstrate that the inversion method also succeeds for these cases in modelling a risk capital requirement that achieves the required probability of solvency in good approximation.

Keywords Parameter uncertainty, parameter risk, Solvency capital requirement

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1 Introduction

The first pillar of the Solvency II project (cf. [11]) requires the quantification of all material risks of an insurance undertaking. Mathematically, the potential losses of the next business year can be described by a random variable $\boldsymbol{X}$. The required risk capital is then given as the 99.5%-quantile of $\boldsymbol{X}$.

Throughout this contribution, we assume that $\boldsymbol{X}$ is equal to $\boldsymbol{X}(\theta_0)$ for a fixed parameter (vector) θ_0. We concentrate on the parameter risk, that means, we assume that $\boldsymbol{X}$ is a member of a parametric distribution family $\mathcal{F} = \{\boldsymbol{X}(\theta) | \theta \in I \subseteq \mathbb{R}^{\cdot}\}$ and that the entity knows $\mathcal{F}$, but can only estimate $\theta_0 \in I \subseteq \mathbb{R}^d$ with $\boldsymbol{X} = \boldsymbol{X}(\theta_0)$ resp. $F_{\boldsymbol{X}} = F_{\boldsymbol{X}(\theta_0)}$ from historical data $x_1, \ldots, x_n$, drawn from the random variable $\boldsymbol{X} = (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n)$ where $\boldsymbol{X}_i \sim \boldsymbol{X}$ and $(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n)$ are independent of $\boldsymbol{X}$. In this setting, the undertaking faces a parameter uncertainty.

We demonstrate the problem of parameter uncertainty from the undertaking's perspective by an example.

Example 1 Consider a set of historical data given as a sample of size $n = 10$ drawn from a normally distributed loss variable $\boldsymbol{X}$:

$$\{98.56; 105.66; 104.80; 109.04; 125.43; 108.50; 105.48; 98.07; 93.99; 107.92\}.$$

The true parameters (μ, σ) are unknown to the undertaking. It estimates the parameters of the distribution using the maximum likelihood method and finds $(\hat{\mu}, \hat{\sigma}) = (105.75, 8.13)$. The 99.5%-quantile of the normally distributed random variable with parameters $(\hat{\mu}, \hat{\sigma})$ is equal to 126.68. Is this an appropriate risk capital taking the parameter uncertainty with respect to μ and σ into account? If not, how should we calculate the risk capital?

If the undertaking does not know the true parameter θ_0, it has to estimate the risk capital. In case the size n of the sample is small, this underestimation can not be avoided in any situation. However, Solvency II does not require to hold an adequate risk capital in any case but only in 99.5% of the cases. More precisely, to apply article 101 of the Solvency II regulation guidelines [11] we need to model the risk capital requirement **SCR** such that it will not be exceeded by the loss $\boldsymbol{X}$ of the next business year with probability 99.5% - taking into account the randomness of $\boldsymbol{X}$ and the randomness of the historical sample $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ (cf. [4], Definition 1).

In this article, we investigate the practical aspects of parameter uncertainty. In Section 2 we recall the definition of the probability of solvency (see also [5], Section 2.1). It measures the underestimation of the risk capital requirement. We explicitly state the probability of solvency for different confidence levels, different distributions and different sample sizes. This helps to assess the impact of the parameter risk in practice.
The inversion method introduced in [4] is an approach to model the risk capital requirement taking parameter uncertainty into account (cf. Subsection

2.3. If $\mathcal{F}$ is a transformed location-scale family and the parameter θ_0 is estimated using either the maximum likelihood method, percentile matching or the Bayesian estimate, the inversion method has been proven to lead to a risk capital meeting the required confidence level. The same holds if $\mathcal{F}$ is location-scale and the parameter (vector) is estimated using the method of moments. However, not all parametric distribution families used in practice are transformed location-scale and in some situations estimation methods different from the ones mentioned above a more popular.
In this contribution, we apply the inversion method to the lognormal distribution (which is not a location-scale family) in the case where the estimation method is the method of moments. Moreover, we consider the two-parameter Gamma distribution (which is not a transformed location-scale family) together with the method of moments and the maximum likelihood method. Our experimental data demonstrate that the inversion method leads to a modelled risk capital achieving the required confidence level in good approximation.

Notation (cf. [4]): Throughout the article all random variables are printed in bold. We define $\boldsymbol{\zeta}$ and $\boldsymbol{\xi}$ as uniformly distributed random variables on $[0;1]^n$ resp. $[0;1]$. By ζ and ξ we denote fixed realizations of these random variables. Let $I \subseteq \mathbb{R}^d$ be a set of parameters and let $\{\boldsymbol{X}(\theta) | \, \theta \in I\}$ and $\{F_{\boldsymbol{X}(\theta)} | \, \theta \in I\}$ be the corresponding set of random variables resp. the set of corresponding distribution functions. We assume that the inverse $F^{-1}_{\boldsymbol{X}(\theta)}$ of the cumulative distribution function of $\boldsymbol{X}$ exists. We define the function $X : [0;1] \times I \to \mathbb{R}$ by $X(\boldsymbol{\xi};\theta) := F^{-1}_{\boldsymbol{X}(\theta)}(\boldsymbol{\xi})$ and use $X(\boldsymbol{\xi};\theta)$ to denote the random variable $\boldsymbol{X}(\theta)$.

2 The probability of solvency and the inversion method

2.1 The probability of solvency

We recall the approach chosen in [4], Section 2.
Given the historical data $(x_1, \ldots, x_n)$ drawn from the random variable $\boldsymbol{X} = (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n)$, whose cumulative distribution function $F_{\boldsymbol{X}} = F_{\boldsymbol{X}(\theta)}$ is known except for the true, but unknown parameter $\theta_0 \in I \subseteq \mathbb{R}^d$, we assume that the undertaking determines its risk capital by the following two step procedure:

1. Using some method M, the undertaking generates a probability distribution $\mathcal{P} = \mathcal{P}(x_1, \ldots, x_n; M)$ for the parameter $\boldsymbol{\theta}_{sim}$ depending on the sample $(x_1, \ldots, x_n)$.
2. The **modelled risk $\boldsymbol{Y}$** is defined by

$$\boldsymbol{Y}(\hat{\theta}) := X(\boldsymbol{\xi}, \boldsymbol{\theta}_{sim}),$$

where $\boldsymbol{\xi}$ is a [0;1]-uniformly distributed random variable and the **modelled risk capital requirement with confidence level** α is set to

$$SCR(\alpha; x_1, \ldots, x_n; M) = F^{-1}_{\boldsymbol{Y}(\hat{\theta})}(\alpha). \tag{1}$$

We define the probability of solvency given a method M resp. the probability distribution $\mathcal{P}$:

Definition 1 Let M be a method to generate a probability distribution $\mathcal{P} = \mathcal{P}(\hat{\theta}; M)$ for the simulated parameter $\boldsymbol{\theta}_{sim}$. The probability

$$P(\boldsymbol{X} \leq SCR(\alpha; \boldsymbol{X}_1, \ldots, \boldsymbol{X}_n; M)) \tag{2}$$

is called the **probability of solvency** for $\alpha \in [0; 1]$. We say that M resp. $\boldsymbol{\theta}_{sim}$ are **appropriate with respect to** α if the probability of solvency for α equals the required confidence level α.
A method M resp. a probability distribution $\boldsymbol{\theta}_{sim}$ is called **appropriate** if it is appropriate for every $\alpha \in (0; 1)$.

2.2 Neglecting parameter uncertainty

The most common way of generating a probability distribution for the simulated parameter $\boldsymbol{\theta}_{sim}$ is to set $\boldsymbol{\theta}_{sim} \equiv \hat{\theta}$. This approach ignores the parameter uncertainty. For Example 1 in Section 1, it yields

$$SCR(\alpha; x_1, \ldots, x_n; without) := F^{-1}_{\boldsymbol{X}(\hat{\mu},\hat{\sigma})}(99.5\%) = 126.68.$$

We determine the probability of solvency

$$P(\boldsymbol{X} \leq SCR(\alpha; \boldsymbol{X}_1, \ldots, \boldsymbol{X}_n; without))$$

where both $\boldsymbol{X}$ and $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ are random using a Monte-Carlo simulation for different confidence levels, different distributions and estimation methods for the case where the sample size n is equal to n (see Table 1 on p. 5). Note that this extends Table 1 in [5].
The results for sample sizes $n = 20$, 50 and 100 are given in the appendix.

Remark 1 In all cases considered the probability of solvency is significantly lower than the required confidence level. The 99.5%-quantile can be interpreted as the event which occurs at most once in 1 out of 200 years. For the normal distribution together with the maximum likelihood estimation insolvency would actually be expected in less than 46 years, for the two-parameter Gamma distribution it is even more likely.
For Example 1 in Section 1 we conclude that a risk capital of 126.68 is not sufficient to cover the 200-year event.
The figures are only slightly better for $n = 20$ (see Table 8 to Table 9 in the appendix). For $n = 20$ almost all distributions have a probability of solvency of less than 99% for the required confidence level of 99.5%. Only for samples of size $n = 50$ the probability of solvency lies above 99%.

Distribution	Estimation method True parameter	95%	99%	99.5%
Gamma	ML $k = 0.5$, $\beta = 1$	91.21%	96.40%	97.40%
	ML $k = 2$, $\beta = 1$	91.41%	96.70%	97.70%
	MM $k = 0.5$, $\beta = 1$	90.42%	95.70%	96.79%
	MM $k = 2$, $\beta = 1$	91.59%	96.79%	97.76%
Normal/ Lognormal (two parameter)	ML*	91.45%	96.77%	97.76%
Normal/ Lognormal	MM $\mu = 0.1$, $\sigma = 0.1$	91.41%	96.74%	97.74%
	MM $\mu = 1$, $\sigma = 0.1$	91.41%	96.74%	97.74%
	MM $\mu = 1$, $\sigma = 1$	89.37%	95.17%	96.44%
Exponential (one parameter)	ML*	92.75%	97.74%	98.58%
Pareto (two parameter)	ML*	91.40%	96.99%	98.02%

Table 1 Solvency probabilities in the case that the risk capital is calculated without taking parameter uncertainty into account for $n = 10$ for different confidence levels, different continuous distributions and different methods of estimation, ML=maximum likelihood, MM=method of moments. For the distributions and estimation methods with * the probability of solvency can be proven to be independent of the chosen parameter (see [5]). The figures have been determined using a Monte-Carlo simulation with 10.000.000 realizations of $\boldsymbol{X}$ and 10.000.000 different samples $\{x_1, \ldots, x_n\}$ of size n to determining $SCR(\alpha; \{x_1, \ldots, x_n\}; M)$.

2.3 Description of the inversion method

The inversion method proposed by Fröhlich and Weng (see [4]) leads to an appropriate probability distribution $\boldsymbol{\theta}_{sim}^{inv}$ in the sense of Definition 1 for transformed location-scale families together with the maximum likelihood method, the percentile matching or the Bayesian estimation method with a certain prior distribution and location-scale families together with the method of moments. In particular, it works for the normal distribution. In the case of Example 1 in Section 1, the inversion method yields an appropriate risk capital requirement of 175.70, an increase by 38% compared to the risk capital requirement not taking parameter uncertainty into account.

Given the random variable $\boldsymbol{X}$ with fixed, but unknown parameter θ, the inversion method consists of two steps:

1. The historical data $(x_1, \ldots, x_n)$ are realizations of the independent, identically distributed random variables $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$, such that $\boldsymbol{X}_i \sim \boldsymbol{X}$ for $i = 1, \ldots, n$. We can write x_i as $F_{\boldsymbol{X}}^{-1}(\zeta_i)$, where $(\zeta_1, \ldots, \zeta_n)$ is a realization of a vector $\boldsymbol{\zeta} = (\boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_n)$ of independent, uniformly distributed ran-

dom variables and F is the distribution function of $\boldsymbol{X}$. Define the function $\mathfrak{h}_\zeta(\theta) := \hat{\theta}(\zeta, \theta)$ where $\hat{\theta}(\zeta, \theta)$ is the estimate of θ depending on θ and the fixed historic observation $(\zeta_1, \ldots, \zeta_n)$. The inversion method defines the probability distribution $\mathcal{P}$ (cf. Subsection 2.1) by

$$\boldsymbol{\theta}_{sim}^{inv} = \boldsymbol{\theta}_{sim}^{inv}(\boldsymbol{\zeta}, \hat{\theta}) := \mathfrak{h}_{\boldsymbol{\zeta}}^{-1}(\hat{\theta}) \tag{3}$$

where $\boldsymbol{\zeta} = (\boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_n)$ is a vector of independent, uniformly distributed random variables.

2. Let $\boldsymbol{X}$ be the true risk. We define the modelled risk (see Equation (1)) as

$$\boldsymbol{Y}(\hat{\theta}) := X\left(\boldsymbol{\xi}, \mathfrak{h}_{\boldsymbol{\zeta}}^{-1}(\hat{\theta})\right) = F^{-1}_{X(\boldsymbol{\theta}_{sim}^{inv})}(\boldsymbol{\xi}). \tag{4}$$

From the two-step procedure above, we define the following algorithm to get realizations θ_{sim}^{inv} of $\boldsymbol{\theta}_{sim}^{inv}$:

1. Draw $\zeta = (\zeta_1, \ldots, \zeta_n)$ from $\boldsymbol{\zeta} = (\boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_n)$, where $\boldsymbol{\zeta}_i$ are uniformly distributed on $[0; 1]$.
2. Solve the equation

$$\hat{\theta}(\xi, \cdot) = \hat{\theta}_0, \tag{5}$$

where $\hat{\theta}_0$ is the estimated parameter given the data $(x_1, \ldots, x_n)$, and set θ_{sim}^{inv} equal to the solution.

Depending on the distribution of $\boldsymbol{X}$ there are different methods to solve Equation (5):

1. In some cases, Equation (5) can be solved **analytically**. For example, if $\boldsymbol{X}$ belongs to a transformed location-scale familie $\mathcal{F} = \{h(\mu + \sigma\boldsymbol{Z}) | \mu \in \mathbb{R}, \sigma > 0\}$ for some fixed random variable $\boldsymbol{Z}$ and if the chosen estimation method is the maximum likelihood method, then

$$\boldsymbol{\mu}_{sim} = \hat{\mu}_0 - \frac{\hat{\mu}(\boldsymbol{Z}_1, \ldots, \boldsymbol{Z}_n)}{\hat{\sigma}(\boldsymbol{Z}_1, \ldots, \boldsymbol{Z}_n)} \cdot \hat{\sigma}_0 \quad \text{and } \boldsymbol{\sigma}_{sim} = \frac{\hat{\sigma}_0}{\hat{\sigma}(\boldsymbol{Z}_1, \ldots, \boldsymbol{Z}_n)}$$

where $\hat{\theta}_0 = (\hat{\mu}_0, \hat{\sigma}_0)$ is the estimate of (μ, σ) for a given observation $(x_1, \ldots, x_n)$ and $\hat{\mu}(\boldsymbol{Z}_1, \ldots, \boldsymbol{Z}_n)$ resp. $\hat{\sigma}(\boldsymbol{Z}_1, \ldots, \boldsymbol{Z}_n)$ are the random variables depending on the vector $(\boldsymbol{Z}_1, \ldots, \boldsymbol{Z}_n)$, with $\boldsymbol{Z}_i \sim \boldsymbol{Z}$, using the maximum likelihood method (cf. [4], Corollary 2).

Example 2 The two-parameter Pareto distribution $\text{Par}(\beta, k)$ with scale parameter β and shape parameter k given by the density function

$$f(x) = k \cdot \frac{\beta^k}{x^{k+1}} \text{ for } x \geq \beta.$$

is a transformed location-scale family derived via the function $h(x) = \exp(x)$ from the generalized exponential distribution. It is very popular for modelling extreme risks. The maximum likelihood estimates for the parameters β and k given the sample $\{x_1, \ldots, x_n\}$ are (cf. [6], Section 5.3)

$$\hat{\beta} = \min_i x_i \text{ and } \hat{k} = \frac{n}{\sum \ln x_i - \ln \hat{\beta}}.$$

Note that $\hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{k}}$ are independent random variables [8]. Moreover, $\hat{\boldsymbol{\beta}}$ is Pareto distributed with scale parameter β and shape parameter $n \cdot k$ and $\hat{\boldsymbol{k}}$ has an inverse gamma distribution with shape parameter $n-1$ and scale parameter $n \cdot k$.
Using the inversion method we get a realization (k_{sim}, β_{sim}) of $(\boldsymbol{k}_{sim}, \boldsymbol{\beta}_{sim})$ by

$$k_{sim} = \frac{\hat{k}}{n} \cdot F^{-1}_{\Gamma(n-1,1)}(\zeta_1) \text{ and } \beta_{sim} = \hat{\beta} \cdot \left(F^{-1}_{\text{Par}(1,n \cdot k_{sim})}(\boldsymbol{\zeta}_2)\right)^{-1}$$

where $F^{-1}_{\Gamma(n-1,1)}$ resp. $F^{-1}_{\text{Par}(1,n \cdot k_{sim})}$ is the inverse cumulative distribution function of the Gamma distribution with shape parameter $n-1$ and scale parameter 1 resp. of the Pareto distribution with shape parameter 1 and scale parameter $n \cdot k_{sim}$W and ζ_1, ζ_2 are realizations of two independent, uniformly distributed random variables $\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2$.
Let us consider two samples of different sizes $n = 10$ and $n = 20$ taken from [7], Exercise 13.57.
The first sample is given by

$$S_1 = \{132; 149; 476; 147; 135; 110; 176; 107; 147; 165\}$$

and the second sample is

$$S_2 = S_1 \cup \{135; 117; 110; 111; 226; 108; 102; 108; 227; 102\}.$$

Using the maximum likelihood method for S_1 (resp. S_2) we obtain

$$\hat{\beta} = 107 \text{ (resp. } \hat{\beta} = 102) \quad \text{and} \quad \hat{k} = 2.5908 \text{ (resp. } \hat{k} = 3.0185).$$

The table below displays the impact of the consideration of parameter uncertainty on the risk capital calculation for both samples. The risk capital with parameter uncertainty has been determined using a Monte-Carlo simulation with 1,000,000 realizations.

Sample	Risk capital		Increase in %
	without param. risk	with param. risk	
S_1	827.03	2,144.73	+159%
S_2	590.07	837.86	+42%

The Pareto distribution is a probability distribution with a heavy tail. Therefore, it is not surprising that the parameter risk is even more relevant than in the case of the normal distribution. Moreover, the result reflects the fact that the parameter risk declines with the size of the sample.

2. In other cases, we have to solve Equation (5) **numerically**. We demonstrate this approach in Section 3 and Section 4.

3. In the case of an univariate distribution where the **confidence interval** of the estimate is known we can use the one-sided confidence interval to construct the distribution $\boldsymbol{\theta}_{sim}^{inv}$.
   If we determine the one-sided confidence interval $I(\alpha;\hat{\theta}) = [B(\alpha;\hat{\theta});\infty)$ for the parameter θ defined by $P(\theta \in I(\alpha,\hat{\theta})) = \alpha$, we can use the lower bound $B(\cdot,\hat{\theta})$ to simulate $\boldsymbol{\theta}_{sim}^{inv}$ setting $\boldsymbol{\theta}_{sim}^{inv}(\hat{\theta}) := B(\boldsymbol{\varsigma};\hat{\theta})$ (cf. [4], Proposition 1).

Example 3 (a) Let $\boldsymbol{X}$ be a $N(\mu,1)$-distributed random variable. The $(1-\alpha)$-confidence interval for the parameter μ is $\left[\hat{\mu} - \frac{\phi^{-1}(\alpha)}{\sqrt{n}}, \infty\right)$, where ϕ is the distribution function of a $N(0;1)$-distributed random variable (see e.g. [6]). Hence,

$$\boldsymbol{\mu}_{sim} \sim \hat{\mu} - \frac{\phi^{-1}(\boldsymbol{\varsigma})}{\sqrt{n}} \sim \hat{\mu} - \frac{\boldsymbol{Z}}{\sqrt{n}},$$

where $\boldsymbol{\varsigma}$ is uniformly distributed on $[0;1]$ and $\boldsymbol{Z}$ is $N(0;1)$-distributed.

(b) The confidence interval approach is a tool that can also be applied to discrete distributions. Let $\boldsymbol{X}$ be a Poisson-distributed random variable with fixed but unknown parameter $\lambda > 0$. Given a sample $x_1, \ldots, x_n$, set $\hat{\lambda} := \frac{1}{n}\sum x_i$.
Using the normal approximation we determine an one-sided $(1-\alpha)$-confidence interval for λ by $\left[\hat{\lambda} - \phi^{-1}(\alpha)\sqrt{\frac{\hat{\lambda}}{n}}, \infty\right)$, where ϕ is the distribution function of the standard normal distribution.
We set then

$$\boldsymbol{\lambda}_{sim} := \hat{\lambda} - \phi^{-1}(\boldsymbol{\varsigma})\sqrt{\frac{\hat{\lambda}}{n}},$$

where $\boldsymbol{\varsigma}$ is uniformly distributed on $[0;1]$.

3 The lognormal distribution with the method of moments

The inversion method explained in Subsection 2.3 has been proved to be appropriate for the lognormal distribution together with the maximum likelihood method (cf. [4]). This estimation method has a drawback: it is biased. For some applications like reserving we prefer an unbiased estimation method such as the method of moments. However, it is not known whether the inversion method is appropriate if we use the lognormal distribution together with the method of moments.

3.1 Application of the inversion method

Lemma 1 *Let $x_1, \ldots, x_n$ be a sample drawn from $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$, where $\boldsymbol{X}_i$ are independent, identically distributed random variables such that $\boldsymbol{X}_i \sim LN(\mu, \sigma)$*

for $i = 1, \ldots, n$. The estimates $\hat{\mu}$ and $\hat{\sigma}$ of the parameters μ and σ using the method of moments are

$$\hat{\mu} = \ln\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) - \frac{\hat{\sigma}^2}{2} \qquad \textit{and} \tag{6}$$

$$\hat{\sigma}^2 = \ln\left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right) - 2\ln\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right). \tag{7}$$

Proof The assertion follows from $E[X] = e^{\mu+\sigma^2/2}$ and $E[X^2] = e^{2(\mu+\sigma^2)}$ (see [6], Chapter 14, Section 3).

□

Note that setting $SCR(\alpha; \boldsymbol{X}_1, \ldots, \boldsymbol{X}_n; without)) := F^{-1}_{\boldsymbol{X}(\hat{\theta})}(\alpha)$, that is, ignoring parameter uncertainty, implies a probability of solvency below the required confidence level (cf. Table 1 on p. 5).

We adapt the inversion method to the lognormal distribution together with the method of moments.

Lemma 2 *Let $x_1, \ldots, x_n$ be a sample drawn from $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$, where $\boldsymbol{D}_i$ are independent, identically distributed random variables such that $\boldsymbol{D}_i \sim LN(\mu, \sigma)$ for $i = 1, \ldots, n$. Let $(\hat{\mu}, \hat{\sigma})$ be the estimates of the true parameters (μ, σ) using the method of methods. A realization $(\mu_{sim}, \sigma_{sim})$ of the distribution $(\boldsymbol{\mu}_{sim}, \boldsymbol{\sigma}_{sim})$ is given as the simultaneous solution of the following two equations*

$$\hat{\sigma}^2 = -\ln\left(\frac{1}{n}\right) - 2\ln\left(\sum_{i=1}^{n} F^{-1}_{LN(0,\sigma_{sim})}(\zeta_i)\right) + \ln\left(\sum_{i=1}^{n} F^{-1}_{LN(0,2\sigma_{sim})}(\zeta_i)\right) \tag{8}$$

$$\mu_{sim} = \hat{\mu} + \frac{1}{2}\hat{\sigma}^2 - \ln\left(\frac{1}{n}\right) - \ln\left(\sum_{i=1}^{n} F^{-1}_{LN(0,\sigma_{sim})}(\zeta_i)\right), \tag{9}$$

where $\zeta_1, \ldots, \zeta_n$ are realizations of independent $[0;1]$-uniformly distributed random variables $\boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_n$ and $F^{-1}_{LN(0,\sigma)}(\cdot)$ is the inverse cumulative distribution function of a lognormal random variable with parameters $\mu = 0$ and σ.

Proof The function $h_\zeta(\mu, \sigma) := (\hat{\mu}, \hat{\sigma})$ on p. 6 is given by

$$h_\zeta(\theta) = \left(\ln\left(\frac{1}{n}\sum_{i=1}^{n} F^{-1}_{\mathrm{LN}(\mu,\sigma)}(\zeta_i)\right) - \frac{\hat{\sigma}^2}{2},\right.$$
$$\left.\left(\ln\left(\frac{1}{n}\sum_{i=1}^{n} F^{-1}_{\mathrm{LN}(\mu,\sigma)}(\zeta_i)_i^2\right) - 2\ln\left(\frac{1}{n}\sum_{i=1}^{n} F^{-1}_{\mathrm{LN}(\mu,\sigma)}(\zeta_i)_i\right)\right)^{\frac{1}{2}}\right).$$

Solving for μ and σ yields

$$\mu = \hat{\mu} + \frac{1}{2}\hat{\sigma}^2 - \ln\left(\frac{1}{n}\right) - \ln\left(\sum_{i=1}^{n} F^{-1}_{\mathrm{LN}(0,\sigma)}(\zeta_i)\right)$$

and

$$\mu = \hat{\mu} + \hat{\sigma}^2 - \frac{1}{2}\ln\left(\frac{1}{n}\right) - \frac{1}{2}\ln\left(\sum_{i=1}^{n} F^{-1}_{\mathrm{LN}(0,2\sigma)}(\zeta_i)\right),$$

which, after some algebraic transformations, can be seen to be equivalent to Equations (8) and (9).

□

Note that Equation (8) can easily be solved using Newton's iterative method. We set $\boldsymbol{Y}(\hat{\theta}) := X(\boldsymbol{\xi}, (\boldsymbol{\mu}_{sim}, \boldsymbol{\sigma}_{sim}))$ and

$$SCR(\alpha; X_1, \ldots, X_n; inv) = F^{-1}_{\boldsymbol{Y}(\hat{\theta})}(\alpha).$$

Table 2 displays the probability of solvency $P(\mathbf{X} \leq SCR(\alpha; \boldsymbol{X}_1, \ldots, \boldsymbol{X}_n; inv))$. The experimental results support the hypothesis that the inversion method leads to a probability of solvency that achieves the required confidence level in good approximation.

n	μ	σ	$\alpha = 95\%$	$\alpha = 99\%$	$\alpha = 99.5\%$
10	0.1	0.1	94.98%	99.01%	99.50%
		1	95.17%	98.99%	99.51%
	1	0.1	94.92%	99.01%	99.49%
		1	95.22%	99.07%	99.52%
20	0.1	0.1	94.95%	99.02%	99.50%
		1	95.27%	99.04%	99.53%
	1	0.1	95.05%	99.01%	99.50%
		1	95.11%	99.05%	99.55%

Table 2 $P(\mathbf{X} \leq SCR(\alpha; \boldsymbol{X}_1, \ldots, \boldsymbol{X}_n; inv))$ for the lognormal distribution using the inversion method and considering different sample sizes n, confidence levels α and values of the true parameters μ and σ determined using a Monte-Carlo simulation with $100,000$ samples of size n of a lognormal random variable $LN(\mu, \sigma)$ and performing $10,000$ realizations of the distribution of $(\boldsymbol{\mu}_{sim}, \boldsymbol{\sigma}_{sim})$ given a fixed sample.

3.2 Example of the impact on risk capital calculation

We investigate the impact of the inversion method on the modelled risk capital considering the sample used in [4], Section 5, drawn from a lognormally distributed loss variable $\boldsymbol{X}$:

$$S_1 = \{150.01; 152.33; 120.47; 131.87; 139.07;\\ 157.97; 128.37; 122.89; 166.47; 133.18\}.$$

In Table 3 we compare the risk capital requirements using the inversion method with those not taking parameter uncertainty into account.

Estimation method	Estimated parameters		Risk capital	
	$\hat{\mu}$	$\hat{\sigma}$	without param. risk	with param. risk
MM	4.9380	0.1054	182.92	204.07
ML (cf. [4], Sect. 5)	4.9380	0.1047	182.65	203.06

Table 3 Comparison of the two estimation methods for sample S_1 using a Monte-Carlo simulation with 1,000,000 realiziations of the distribution of $(\boldsymbol{\mu}_{sim}, \boldsymbol{\sigma}_{sim})$

Table 4 displays the results for the more volatile sample

$$S_2 = \{150.01; 182.10; 120.47; 211.50; 139.07; \\ 157.97; 199.35; 122.89; 166.47; 133.18\}.$$

Estimation method	Estimated parameters		Risk capital	
	$\hat{\mu}$	$\hat{\sigma}$	without param. risk	with param. risk
MM	5.0470	0.1868	251.84	307.97
ML (cf. [4], Sect. 5)	5.0471	0.1856	250.93	302.90

Table 4 Comparison of the two estimation methods for sample S_2 using a Monte-Carlo simulation with 1,000,000 realiziations of the distribution of $(\boldsymbol{\mu}_{sim}, \boldsymbol{\sigma}_{sim})$

In both cases, the risk capital reflecting parameter uncertainty is significantly higher than the one calculated without taking parameter uncertainty into account. The results obtained using each of the two estimation methods are of the same magnitude, but we observe that the maximum likelihood method leads to a slightly lower risk capital requirement.

4 The gamma distribution

Let $\boldsymbol{X}$ be a gamma distributed random variable with shape parameter k and scale parameter β with density function

$$f(x; k, \beta) = \frac{\left(\frac{1}{\beta}\right)^k \cdot x^{k-1} \exp(-\frac{x}{\beta})}{\Gamma(k)}.$$

As mentioned before, the two-parameter gamma distribution is not a transformed location-scale family. It is therefore not known whether the inversion method is appropriate in this case.

Lemma 3 *Let $x_1, \ldots, x_n$ be a sample drawn from $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$, where the $\boldsymbol{X}_i$ are independent, identically distributed random variables such that $\boldsymbol{X}_i \sim \Gamma(k, \beta)$ for $i = 1, \ldots, n$. The estimates $(\hat{k}, \hat{\beta})$ of the true parameters (k, β) using the method of moments are given by*

$$\hat{k} = \frac{\overline{d}_n^2}{S_n^2} \text{ and } \hat{\beta} = \frac{S_n^2}{\overline{d}_n} = \frac{\overline{d}_n}{\hat{k}}, \tag{10}$$

where $\overline{d}_n$ is the sample mean and S_n^2 is the sample variance.

Proof This follows from $E[X] = k \cdot \beta$ and $\mathrm{Var}[X] = k \cdot \beta^2$ (cf. [6], Chapter 17). □

The maximum likelihood estimation is more complicated: For a random sample $x_1, \ldots, x_n$ taking the derivative of the logarithm of the likelihood function yields

$$\ln(\hat{k}) - \psi(\hat{k}) = y \text{ and } \hat{k} \cdot \hat{\beta} = A$$

with $y = \ln(A/G)$ where $A = \frac{1}{n} \sum x_i$ is the arithmetic and $G = \sqrt[n]{\prod x_i}$ the geometric mean (cf. [3], Chapter 2) and ψ is the digamma function. Note that y is always strictly positive, unless $x_1 = \ldots = x_n = 0$. Since y is independent of the scale factor β, the distribution of $\hat{\boldsymbol{k}}$ depends only on k and the sample size.
for $y \neq 0$, the solution of the equation for $\hat{k}$ can be found using an iteration (cf. [3], Chapter 2)

$$k_m = \frac{k_{m-1} \left(\ln(k_{m-1}) - \psi(k_{m-1})\right)}{y}$$

seeded with

$$k_0 = \frac{1 + \sqrt{1 + 4y/8}}{4y}.$$

4.1 Inversion method with the method of moments

We apply the inversion method to the two-parameter gamma distribution together with the method of moments.

Lemma 4 *Let $x_1, \ldots, x_n$ be a sample drawn from $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$, where $\boldsymbol{X}_i$ are independent, identically distributed random variables such that $\boldsymbol{X}_i \sim \Gamma(k, \beta)$ for $i = 1, \ldots, n$. Let $(\hat{k}, \hat{\beta})$ be the estimates of the true parameters (k, β) using the method of methods. A realization (k_{sim}, β_{sim}) of the parameter distribution $(\boldsymbol{k}_{sim}, \boldsymbol{\beta}_{sim})$ can be constructed as follows: Choose realizations $\zeta_1, \ldots, \zeta_n$ of independent, on $[0;1]$-uniformly distributed random variables $\boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_n$ and determine a zero k_{sim} of the function*

$$f(x) = \hat{k} - \frac{\overline{X}_n^2(\zeta_1, \ldots, \zeta_n; x)}{\frac{1}{n-1} \sum_{i=1}^{n} \left(F_{\Gamma(x,1)}^{-1}(\zeta_i) - \overline{X}_n(\zeta_1, \ldots, \zeta_n; x)\right)^2},$$

where $\overline{X}_n(\zeta_1,\dots,\zeta_n;x) := \frac{1}{n}\sum_{i=1}^n F^{-1}_{\Gamma(x,1)}(\zeta_i)$*. Then set*

$$\beta_{sim} = \frac{n\cdot\hat{k}\cdot\hat{\beta}}{\sum_{i=1}^n F^{-1}_{\Gamma(k_{sim},1)}(\zeta_i)}.$$

Proof The function $h_\zeta(k,\beta) := (\hat{k},\hat{\beta})$ on p. 6 is given by

$$h_\zeta(k,\beta) := (\hat{k},\hat{\beta}) = \left(\frac{\overline{X}_n^2(\zeta_1,\dots,\zeta_n;k;\beta)}{\frac{1}{n-1}\sum_{i=1}^n\left(F^{-1}_{\Gamma(k,\beta)}(\zeta_i)-\overline{X}_n(\zeta_1,\dots,\zeta_n;k;\beta)\right)^2},\right.$$
$$\left.\frac{\frac{1}{n-1}\sum_{i=1}^n\left(F^{-1}_{\Gamma(k,\beta)}(\zeta_i)-\overline{X}_n(\zeta_1,\dots,\zeta_n;k;\beta)\right)^2}{\overline{X}_n(\zeta_1,\dots,\zeta_n;k;\beta)}\right).$$

where $\overline{X}_n(\zeta_1,\dots,\zeta_n;k;\beta) := \frac{1}{n}\sum_{i=1}^n F^{-1}_{\Gamma(k,\beta)}(\zeta_i)$. Using

$$\frac{\overline{X}_n^2(\zeta_1,\dots,\zeta_n;k;\beta)}{\sum_{i=1}^n\left(F^{-1}_{\Gamma(k,\beta)}(\zeta_i)-\overline{X}_n(\zeta_1,\dots,\zeta_n;k;\beta)\right)^2} = \frac{\overline{X}_n^2(\zeta_1,\dots,\zeta_n;k;1)}{\sum_{i=1}^n\left(F^{-1}_{\Gamma(k,1)}(\zeta_i)-\overline{X}_n(\zeta_1,\dots,\zeta_n;k;1)\right)^2}$$

and

$$\hat{\beta} = \frac{\overline{X}_n(\zeta_1,\dots,\zeta_n;k;\beta)}{\hat{k}} = \frac{\beta\cdot\sum_{i=1}^n F^{-1}_{\Gamma(k,1)}(\zeta_i)}{n\cdot\hat{k}}$$

the assertion follows.

□

Consequently, we can now set

$$SCR(\alpha;x_1,\dots,x_n;inv) := F^{-1}_{\boldsymbol{Y}(\hat{k},\hat{\beta})}(\alpha),$$

where $\boldsymbol{Y}(\hat{k},\hat{\beta}) = X(\boldsymbol{\xi},(\boldsymbol{k}_{sim},\boldsymbol{\beta}_{sim}))$. Table 6 displays the solvency probabilities $P(\mathbf{X}\le SCR(\alpha;\boldsymbol{X}_1,\dots,\boldsymbol{X}_n;inv))$.

n	β	k	$\alpha = 95\%$	$\alpha = 99\%$	$\alpha = 99.5\%$
10	1	1	95.21%	90.00%	99.53%
		2	95.11%	99.00%	99.49%
		4	95.03%	99.03%	99.52%
	5	1	95.16%	90.05%	99.51%
		2	95.13%	99.01%	99.51%
		4	95.04%	99.02%	99.53%
	10	1	94.92%	90.00%	99.50%
		2	95.08%	99.06%	99.52%
		4	95.06%	99.02%	99.51%
20	1	1	95.12%	90.03%	99.52%
		2	95.09%	99.05%	99.54%
		4	95.10%	99.04%	99.51%
	5	1	95.15%	90.09%	99.56%
		2	95.00%	99.00%	99.50%
		4	95.00%	99.05%	99.52%
	10	1	95.08%	90.06%	99.50%
		2	95.16%	99.05%	99.55%
		4	95.07%	99.00%	99.49%

Table 5 $P(\mathbf{X} \leq SCR(\alpha; \boldsymbol{X}_1, \ldots, \boldsymbol{X}_n; inv))$ for the two-parameter gamma distribution using the inversion method and considering different sample sizes n, confidence levels α and values of the true parameters k and β determined using a Monte-Carlo simulations with $100,000$ samples of size n of a gamma-distributed random variable $\Gamma(k, \beta)$ and performing $10,000$ realizations of the distribution of $(\boldsymbol{k}_{sim}, \boldsymbol{\beta}_{sim})$.

The empirical results support the hypothesis that the inversion method leads to a probability of solvency close to the desired confidence level.

4.2 Inversion method with the maximum likelihood method

We now consider the more popular maximum likelihood method. Unfortunately, the parameter estimation using the maximum likelihood method is rather complicated. For the application of the inversion method we use a simple approximation by Thom [12] which gives good results in practice: Consider the equation for $\hat{k}$

$$\ln(\hat{k}) - \psi(\hat{k}) = y. \tag{11}$$

Using the asymptotic expansion

$$\psi(x) = \log x - 1/(2x) - \sum_{k=1}^{m} (-1)^{k-1} B_k / (2k\gamma^{2k}) + R_m$$

where B_k are the Bernoulli numbers with $B_1 = 1/6$ and $B_2 = 1/30$ etc. and R_m is the remainder after m terms, we derive the approximation

$$\psi(x) \approx \ln(x) - 1/(2x) - 1/(12x^3).$$

Substituting in (11) yields the quadratic equation

$$16y\hat{k}^2 - 6\hat{k} - 1 \approx 0$$

whose only relevant root is

$$\hat{k} = \frac{1 + \sqrt{1 + 4y/3}}{4y}.$$

Note that y depends on the real parameter k.
For the application of the inversion method we first determine realizations $\zeta_1, \ldots, \zeta_n$ of independent, on $[0; 1]$ uniformly distributed random variables $\boldsymbol{\zeta}_n, \ldots, \boldsymbol{\zeta}_n$. A realization k_{sim} of $\boldsymbol{k}_{sim}$ is then given by a root of the function

$$f(x) = \frac{1 + \sqrt{1 + 4y(x)/3}}{4y(x)} - \hat{k}$$

where

$$y(x) = A/G \text{ with } A = \frac{1}{n}\sum x_i, \quad G = \sqrt[n]{\prod x_i}, \quad x_i = F^{-1}_{\Gamma(x,1)}(\zeta_i).$$

Given k_{sim} a realization β_{sim} of $\boldsymbol{\beta}_{sim}$ is determined by

$$\beta_{sim} = \frac{n \cdot \hat{k} \cdot \hat{\beta}}{\sum_{i=1}^{n} F^{-1}_{\Gamma(k_{sim},1)}(\zeta_i)}.$$

Finally, we set

$$SCR(\alpha; x_1, \ldots, x_n; inv) := F^{-1}_{\boldsymbol{Y}(\hat{k},\hat{\beta})}(\alpha) \text{ where } \boldsymbol{Y}(\hat{k}, \hat{\beta}) = X(\boldsymbol{\xi}, (\boldsymbol{k}_{sim}, \boldsymbol{\beta}_{sim})).$$

Table 6 displays probability of solvency $P(\mathbf{X} \leq SCR(\alpha; \boldsymbol{X}_1, \ldots, \boldsymbol{X}_n; inv))$. The empirical results support the hypothesis that the inversion method together with the maximum likelihood method leads to a risk capital requirement that achieves the desired confidence level in good approximation.

4.3 Example of the impact on the risk capital calculation

Let us consider the following sample of size $n = 10$ taken from [7], Exercise 13.14:

$$\{1,500; 6,000; 3,500; 3,800; 1,800; 5,500; 4,800; 4,200; 3,900; 3,000\}. \quad (12)$$

Using the method of moments we obtain

$$\hat{k} = 6.86 \quad \text{and} \quad \hat{\beta} = 553.22$$

Table 7 displays the impact of the consideration of parameter uncertainty on the risk capital calculation. The risk capital with parameter uncertainty has been determined using a Monte-Carlo simulation with 1,000,000 realizations.

n	β	k	$\alpha = 95\%$	$\alpha = 99\%$	$\alpha = 99.5\%$
10	1	1	95.07%	99.04%	99.55%
		2	95.00%	99.01%	99.53%
		4	95.13%	99.06%	99.52%
	5	1	95.05%	98.97%	99.49%
		2	95.09%	99.04%	99.52%
		4	95.00%	99.01%	99.52%
	10	1	94.95%	99.00%	99.48%
		2	94.91%	99.01%	99.53%
		4	95.04%	99.02%	99.52%
20	1	1	95.01%	89.99%	99.50%
		2	95.00%	99.02%	99.51%
		4	95.07%	99.01%	99.56%
	5	1	95.05%	99.00%	99.48%
		2	94.98%	98.97%	99.49%
		4	95.05%	98.98%	99.50%
	10	1	95.05%	99.07%	99.55%
		2	94.95%	99.03%	99.50%
		4	94.87%	98.97%	99.50%

Table 6 $P(\mathbf{X} \leq SCR(\alpha; \boldsymbol{X}_1, \ldots, \boldsymbol{X}_n; inv))$ for the two-parameter gamma distribution using the inversion method and considering different sample sizes n, confidence levels α and values of the true parameters k and β taking $100,000$ samples of size n of a gamma-distributed random variable $\Gamma(k, \beta)$ and performing $10,000$ realizations of the distribution of $(\boldsymbol{k}_{sim}, \boldsymbol{\beta}_{sim})$ given a fixed sample.

Estimation method	without the consideration of parameter risk	with the consideration of parameter risk	Increase in %
MM	8,554.93	11,113.24	+29,90%
ML	8,790.90	11,746.60	+33,62%

Table 7 Required risk capital for the sample given by Equation 12 using method of moments (MM) and maximum likelihood (ML) with and without the consideration of parameter uncertainty

5 Summary and Outlook

This article deals with practical aspects of parameter uncertainty in the context of risk capital calculations.
For a practitioner it is first necessary to assess the impact of parameter uncertainty. In Table 1 we give the probabilities of solvency for risk capital calculations ignoring parameter uncertainty for commonly used distribution families and estimation methods. For all distributions and all estimation methods considered, the probability of solvency is significantly lower than the given confidence level. In some cases, like e.g. for the Gamma distribution and sample size $n = 10$, it leads to a probability of insolvency which is five times higher than required.
Next we recall the inversion method introduced in [4] and explain its use for

different distributions (see Section 2).
We apply the inversion method in two relevant cases which have not been considered so far but are commonly used in practice: the lognormal distribution together with the method of moments (Section 3) and the Gamma distribution with both, the method of moments and maximum likelihood method (Section 4). For both distributions we give experimental results supporting the hypothesis that the inversion method leads to probabilities of solvency achieving the required confidence level in good approximation.
Together with the results derived in [4] the inversion method has proved to be an appropriate and practical tool for modelling parameter uncertainty in risk capital calculations.
In the future, the challenge lies in the consideration of aggregate distributions where the overall risk can be written as the sum of random variables resp. depends on the several random variables, but the historical data are given on a more granular level. An example is the collective risk model where the overall risk is given by $\boldsymbol{S} = \sum_{i=1}^{\boldsymbol{N}} \boldsymbol{X}_i$, $\boldsymbol{X}_i \sim \boldsymbol{X}$, and where we have historical data for the number of claims (i.e. realizations of $\boldsymbol{N}$) and for the amount of the claims (i.e. realizations of $\boldsymbol{X}$).

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The experimental results have been generated using Java and Matlab programs. The calculations are quite time consuming. Therefore, we are very grateful for the opportunity to run the program on the bwGriD cluster of the Hochschule Esslingen.

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7 Appendix

We give the solvency probabilities for risk capital calculations without taking parameter uncertainty into account for sample sizes $n = 20$, $n = 50$ and $n = 100$.

n	Distribution	Estimation method True parameter	95%	99%	99.5%
20	Gamma	ML $k = 0.5,\ \beta = 1$	93.16%	97.86%	98.63%
		ML $k = 2,\ \beta = 1$	93.27%	98.00%	98.76%
		MM $k = 0.5,\ \beta = 1$	92.60%	97.41%	98.26%
		MM $k = 2,\ \beta = 1$	93.26%	97.96%	98.72%
	Normal/ Lognormal (two parameter)	ML*	93.29%	98.03%	98.79%
	Normal/ Lognormal	MM $\mu = 0.1,\ \sigma = 0.1$	93.26%	98.02%	98.78%
		MM $\mu = 1,\ \sigma = 0.1$	93.26%	98.02%	98.78%
		MM $\mu = 1,\ \sigma = 1$	91.81%	97.00%	97.98%
	Exponential (one parameter)	ML*	93.91%	98.42%	99.10%
	Pareto (two parameter)	ML*	93.29%	98.14%	98.90%

Table 8 Solvency probabilities in the case that the risk capital is calculated without taking parameter uncertainty into account for $n = 20$ and different confidence levels, different continuous distributions and different methods of estimation.

n	Distribution	Estimation method True parameter	95%	99%	99.5%
50	Gamma	ML $k = 0.5$, $\beta = 1$	94.28%	98.58%	99.20%
		ML $k = 2$, $\beta = 1$	94.32%	98.64%	99.25%
		MM $k = 0.5$, $\beta = 1$	93.99%	98.37%	99.03%
		MM $k = 2$, $\beta = 1$	94.38%	98.59%	99.21%
	Normal/ Lognormal (two parameter)	ML*	94.33%	98.65%	99.26%
	Normal/ Lognormal	MM $\mu = 0.1$, $\sigma = 0.1$	94.32%	98.64%	99.25%
		MM $\mu = 1$, $\sigma = 0.1$	94.32%	98.64%	99.25%
		MM $\mu = 1$, $\sigma = 1$	93.45%	98.09%	98.84%
	Exponential (one parameter)	ML*	94.54%	98.79%	99.35%
	Pareto (two parameter)	ML*	94.34%	98.69%	99.30%
100	Gamma	ML $k = 0.5$, $\beta = 1$	94.64%	98.80%	99.36%
		ML $k = 2$, $\beta = 1$	94.66%	98.83%	99.38%
		MM $k = 0.5$, $\beta = 1$	94.48%	98.68%	99.27%
		MM $k = 2$, $\beta = 1$	94.64%	98.80%	99.36%
	Normal/ Lognormal (two parameter)	ML*	94.67%	98.83%	99.39%
	Normal/ Lognormal	MM $\mu = 0.1$, $\sigma = 0.1$	94.66%	98.83%	99.38%
		MM $\mu = 1$, $\sigma = 0.1$	94.66%	98.83%	99.38%
		MM $\mu = 1$, $\sigma = 1$	94.10%	98.48%	99.13%
	Exponential (one parameter)	ML*	94.74%	98.89%	99.42%
	Pareto (two parameter)	ML*	94.67%	98.85%	99.40%

Table 9 Solvency probabilities in the case that the risk capital is calculated without taking parameter uncertainty into account for $n = 50$ and $n = 100$ and different confidence levels, different continuous distributions and different methods of estimation.