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Modelling SARS-CoV-II impact

Guillem Barrabés Roura

Modelling the SARS-CoV-II impact

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Author: Guillem Barrabés Roura

Tutor: Dra. Maria Dolors Gil Domènech

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Abstract

"I know that history is going to be dominated by an improbable event. I just don't know what that event will be"

– Nassim Nicholas Taleb, *The Black Swan (2007)*

SARS-CoV-II has affected all of us, the aim of this project was to provide a mathematical model to understand the complex reality and merging two powerful branches of mathematics stochastic processes and dynamical systems. By implementing the model it will be able to show us the dynamics of the current situation and enable policymakers as well as economic agents to understand the current complex reality. This model it is fully designed to be able to provide qualitative valuable information that will be transformed into better decisions and hopefully less demographic as well as economic impact.

Acknowledgements

After an intense period of 6 months, today is the day in which I am finally writing this section of gratitude so as I have finished my final degree project. It has been a period in which I have intensively been learning, not only in the mathematical and economic field, but also on a personal level. Writing this thesis has had a great impact on me and that is why I would like to thank all those people who have kindly helped me throughout this challenging process.

Firstly, I would like to thank my tutor, Maria Dolors Gil Domènech for all her valuable and determining help. You have definitely provided me with all the necessary tools to complete my final degree project satisfactorily..

Secondly, I would particularly like to thank all my professors who have supported me enormously and have always been there to help me when I needed it. And have made me grow not only as a professional but also as a person.

Finally, I would also like to thank my parents and friends for their wise advice and understanding. You have always been there for me too.

Guillem Barrabes Roura
Barcelona, 1st May 2021.

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Introduction

> *"The best moments in our lives are not the passive, receptive, relaxing times... The best moments usually occur when a persons body or mind is stretched to its limits in a voluntary effort to accomplish something difficult and worthwhile."*
>
> – Mihaly Csikszentmihalyi, *Flow (1990)*

I strongly believe that all humans have the responsibility of leaving a positive footprint during our lives. This unexpected pandemic is costing us millions of lives, suffering, and has made us question the habits that define us as humans. Sometimes is when things don't go as expected that the world shifts not only to a new way of thinking but also to a new way of living: the so-called "new normality". The most vulnerable individuals in our society are the ones that are suffering the most dramatic situations during these days. I could not sleep in peace by thinking that, I have done nothing to attempt to mitigate the effects of this sudden shock to our generations, and have not tried to improve peoples lives and opportunities as well as learning of our mistakes by making sure that the future generations will take way better decisions than us.

There are infinite finite ways of describing reality, mathematics is one of them. Mathematical modeling has proved to be one of the most rigorous among them with the ability to provide answers to numerous questions and helping us understand the complex reality. With applications in most fields of the knowledge from physics to religion: a clear example is the recurring uses of the numbers 7, 12, and 40 in the Bible. In this case, the model attempting to build is to help understand a system and to study the behaviour of the different components integrating it and being able to forecast the behavior of the previously described system.

Economics is the science that studies human decision-making in the face of scarcity. These decisions can range from a personal decision to a global policy impacting millions of people across the globe. Every single action and decision made by every single human at every point in time has an economic impact, these decisions and actions range from reading a final degree thesis, not being focused on managing a vessel during the cross the Suez canal to Myanmar's military conflict. To understand the impact of these decisions economics is not only important but crucial. Economics does not dictate the answer, but it can illuminate the different choices.

One of the sources of a market failure is the asymmetry of information, the U.S. SEC Rule 10b5 states that the use of non-public information during trading is considered a breach of fair trading laws. The reason for this is because the information is crucial and it matters. By being able to model a system, users can predict the behavior and forecast future information and this is determinant especially on the decision making under scarcity.

1.1 Origins and motivation

December 2019 I was in Seoul (South Korea) ending my exchange program. On my last days there it seemed that there was only one thing happening on the world: the so called coronavirus. This word was on the mouth and head of all of us. From basic resources scarcity to lock-downs. By that time I only had two questions on my head: How this could end up and how will affect this. Science that day I could not find answers to both questions. On my final dissertation, I attempt to get closer to the answer of both questions and help people as well as decision makers too by developing a mathematical model based on previous academic research.

This semester I am completing the double degree in: business administration and industrial production engineering. During this stage of my life I learned uncountable knowledge and lived memorable experiences. Now that the end of this path is approaching, I thought to bring to life and apply what I consider the most relevant lesson learned here at Universitat Internacional de Catalunya which is: that the the most important thing on life is that we must do our best to add value to people's lives and by doing this future generations will live better than us, and we must also thank our ancestors for doing the same for us so now we can live the way we live, and generation by generation the world will become a better place to live. Considering the situation that we are living know I could not thought of a better topic for my work.

1.2 Aims and scope

The milestone it is not only to learn from first class academic's literature, and acquire knowledge from them. Having the opportunity to expand my knowledge too, such that will allow me to model such a complex situation and presenting it on a way that the behaviour can be seen and understood. Hopefully better information will yield to better decision making and awareness. Improving peoples lives that is the ultimate reason why I devoted so many effort on bringing the project forward and have gave the best of me.

The scope of the project, is that the model must be representative as well as logically rigorous such that can be used in benefit for decision makers as well as policy makers. I know that the idea is challenging technically as well as heavy in terms of work load but the size of the value that will bring to be able to understand the behaviour of the current situation, will for sure allow policymakers drive better the pandemic and minimise the impact.

1.3 Structure of the document

The document starts with an introduction where the frame of the thesis is defined with the origins as well as the motivation as well as the structure. After the introduction, the most relevant literature is commented remarking points of previous investigations and reports written by academics ranging several areas of the knowledge and from different disciplines. On this section a brief reference to all the key points of the articles that lead me to develop this research and helped me shape a solution.

The methodology chapter is divided into four parts: the first part, called preliminary concepts, attempts to define mathematical theorems used in the model and determinant for understanding it.

After this section the model as well as the data collection and the data analysis is well explained on the following sections.

Results chapter is divided into three crucial aspects of the result. From the shape of the equilibrium of the model equations, to connection to existing pure mathematics concepts as well as constants.

Chapter 2

Literature Review

"If I have seen further, it is by standing on the shoulders of giants."

– Isaac Newton, *(1675)*

Since the outbreak of the global pandemic, many academics across the globe started publishing papers and researching on covid. Most of them attempted to contribute to society by working on understanding what was happening and where could bring us to as a society, including me on my final thesis. In this chapter, I write what I consider the papers and articles that I consider that was the starting point that helped me to achieve the result and to draw the conclusions and inspired me with their convincing approaches.

2.1 Economic impact of natural disasters

Reading the papers co-authored by Yasuhide Okuyama, it helped me to understand the economic impact of a natural disaster. He made a simile that helped me understand what the economic impact of a sudden shock to the world was. Okuyama (2004) said: "an economy is like a tennis ball; the harder you throw the ball against a wall, the harder the ball bounces back to you" [Okuyama and Chang, 2004, Okuyama and Sahin, 2009]." A natural disaster throws an economy against a wall; then, how far an economy bounces back depends on the elasticity of the ball, i.e. the resilience of the economy. Knowing the disaster impacts is analogous to understanding how hard the ball (economy) is crushed against the wall" [Okuyama and Chang, 2004]. Some researchers, for example in the case of Albala-Bertrand , he argue that: "since the ball (economy) bounces back anyway, it is unimportant to know how hard the ball is crushed. However, without knowing how the ball (economy) is crushed, the relief efforts may become inefficient and ineffective and the pace of recovery may turn out to be slower. At the same time, if the disasters occur frequently and repeatedly, the ball (economy) accumulates fatigue and the resilience may deteriorate. This will result in the long-run impacts on the economy" [Albala-Bertrand et al., 1993]. This can be consider as a really good simile to conclude their extensive research on modelling shocks on economy. It is important to know how great the shock is and how vulnerable an economy is to such shock and the ability to response to the shock. This is determinant in order to be able to quantify the impact of such shock.

The relationship between disaster impacts and development is also a concern. According to the experienced researcher on the topic, Albala-Bertrand: "Most empirical studies with cross-country data investigating the relationship between development level and disaster impacts conclude that correlation between them is negative" [Albala-Bertrand et al., 1993]. I also would like to remark that according to the same author; "the higher the level of development, the smaller both the number of deaths, injured, and deprived, and the relative material losses" [Albala-Bertrand et al., 1993]. This appears consistent with the disaster theory that as countries develop and grow, they should have sufficient resources, such as financial and/or technological ones, to better manage disaster risk through the implementation of countermeasures and to better manage the adverse impact of disasters.

I would also like to make reference to the well detailed research made by Skidmore and Toya on (2009), on which they both where able to arrive at the following conclusion:" The long-run relationships among disasters, capital accumulation, total factor productivity, and economic growth. The cross-country empirical analysis demonstrates that higher frequencies of climatic disasters are correlated with higher rates of human capital accumulation, increases in total factor productivity, and economic growth"[Skidmore and Toya, 2002]. later on the article the authors say that: "Though disaster risk reduces the expected rate of return to physical capital, risk also serves to increase the relative return to human capital. Thus, physical capital investment may fall, but there is also a substitution toward human capital investment. Disasters also provide the impetus to update the capital stock and adopt new technologies, leading to improvements in total factor productivity"[Skidmore and Toya, 2002] .

2.2 Markov chain and the SIR model

In 1926 McKendrick studied a stochastic epidemic model and found a method to compute the probability for an epidemic to reach a certain final size. He also discovered the partial differential equation governing age-structured populations in a continuous-time framework. In 1927 Kermack and McKendrick studied a deterministic epidemic model and obtained an equation for the final epidemic size, which emphasizes a certain threshold for the population density. Large epidemics can occur above but not below this threshold. The SIR model developed by the both authors and well explained on the following papers, (Kermack and McKendrick)[Kermack and McKendrick, 1927] is the most commonly used in epidemiology. The model divides the population into three different groups: Susceptible, Infected, and Recovered (S,I,R). This model is deterministic and is usually formulated as a system of differential equations. It assumes that the population size is large and is differentiable in terms of time using the equations:

$$\frac{dS}{dt} = -\beta SI \tag{2.1}$$

$$\frac{dI}{dt} = \beta SI - \gamma I \tag{2.2}$$

$$\frac{dR}{dt} = \gamma I \tag{2.3}$$

Where:

- S= Susceptible
- I= Infected
- R= Recovered or Removed
- β=Transmission Rate
- γ= Removal rate

Authors were able to show that it admits a stationary solution, as long as the supply of susceptible individuals is sufficiently large. This model [Kermack and McKendrick, 1927, Kermack and McKendrick, 1932], is difficult to analyze in its full generality, and a number of open questions remain regarding its complex dynamics.

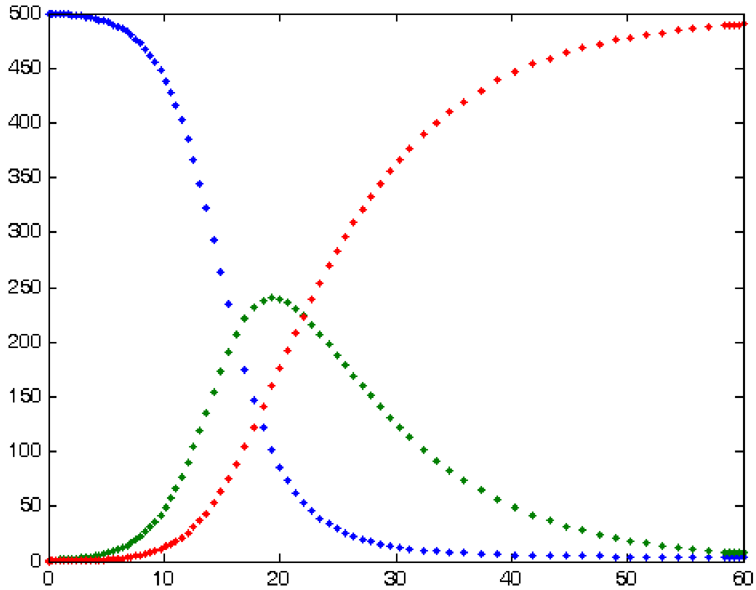

Figure 2.1: S.I.R. epidemic model plot

2.3 Simple mathematical models with complex dynamics

On the 10th of June 1976 The famous mathematician Robert M. May, write an article about extremely simple mathematical equations with very complex dynamic. He concluded that: "first-order difference equations arise in many contexts in the biological, economic, and social sciences. Such equations, even though simple and deterministic, can exhibit a surprising array of dynamical behavior, from stable points to a bifurcating hierarchy of stable cycles, to random fluctuations. There are consequently many fascinating problems, some concerned with delicate mathematical aspects of the fine structure of the trajectories, and some concerned with the practical implications and applications"s [May and Oster, 1976].

On the 28th January 2000, Earn, Rohani, published an article about: simple models for complex dynamical transitions in epidemics. the article stated that: "dramatic changes in patterns of epidemics have been observed throughout this century. For childhood infectious diseases such as measles, the major transitions are between regular cycles and irregular, possibly chaotic epidemics, and from regionally synchronized oscillations to complex, spatially incoherent epidemics. A simple model can explain both kinds of transitions as the consequences of changes in birth and vaccination rates. Measles is a natural ecological system that exhibits different dynamical transitions at different times and places, yet all of these transitions can be predicted as bifurcations of a single nonlinear model" [Earn et al., 2000]. Bothe Earn and Rohani concluded that simple models such as composed by non-linear equations with negative feedback loop, are able to model these events.

2.4 The butterfly effect and the essence of chaos

Edward Lorenz,on 1972 he titled an article: "Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?".[Lorenz, 1972] What Edward Lorenz meant by this statement is that on the one hand minuscule differences in the initial conditions, in the long run, can evolve into two situations differing as much as the presence of a tornado. On the other hand, he questioned our ability to predict events in the long run. He added: "the behavior of the atmosphere is unstable concerning perturbations of small amplitude"[Lorenz, 1972]. This case evolved into what today is called Chaos which is a hyped and trendy branch of mathematics and it was triggered when Lorenz introduced what is called: The butterfly effect. The butterfly effect it is used as a concept on many branches of the knowledge. In Businesses it is used referring that tiny choices that seem insignificant can have huge consequences in the long run, proving a high sensitivity on initial conditions.

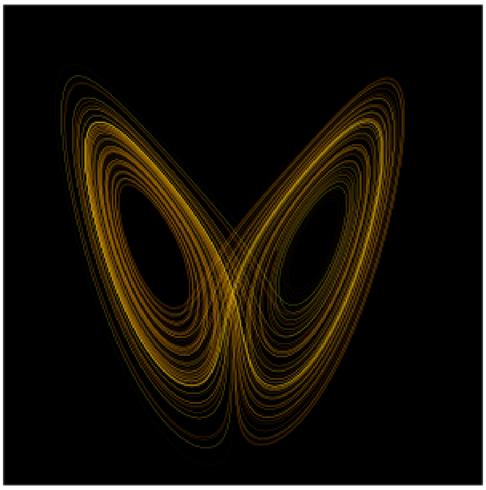

Figure 2.2: Lorenz system with the butterfly shape

In such non-linear dynamical systems, are both deterministic and unpredictable at the same time. That is why some mathematicians use the term pseudo random to define its behaviour. If same initial conditions same results but a tiny little divergence and the results are not comparable. The only way that that can happen is by fractals.

2.5 Is it worth to model a global Pandemic?

Cambridge Econometrics developed several models for European institutions including the European commission including the "E3ME" that include sector desegregation. Recently the famous Hector Pollitt, Head of modeling and chief economist from Cambridge Econometrics posted what many considered their conclusions on trying to make an economic model for the covid impact. The article authored by Pollitt on 2021 concluded that "it is useless to attempt to develop a model that estimates the impact on the GDP"[Pollitt, 2020] and the following: "they are based on things we cannot know. The range of uncertainty in the inputs (e.g. infection rates, fatality rates) will not go away. The economic restructuring that we are likely to see (e.g. a long-term shift to video-conferencing) is also not possible to predict"[Pollitt, 2020]. He left an investigation line that I considered that had potential: "If based on realistic assumptions, models can be a tool to aid understanding and to assist with future planning. In the future, network-based models might be able to provide a richer set of analyses"[Pollitt, 2020]. This thread recently leaved by such an expert on modelling it is the starting point to the model that I attempted to develop.

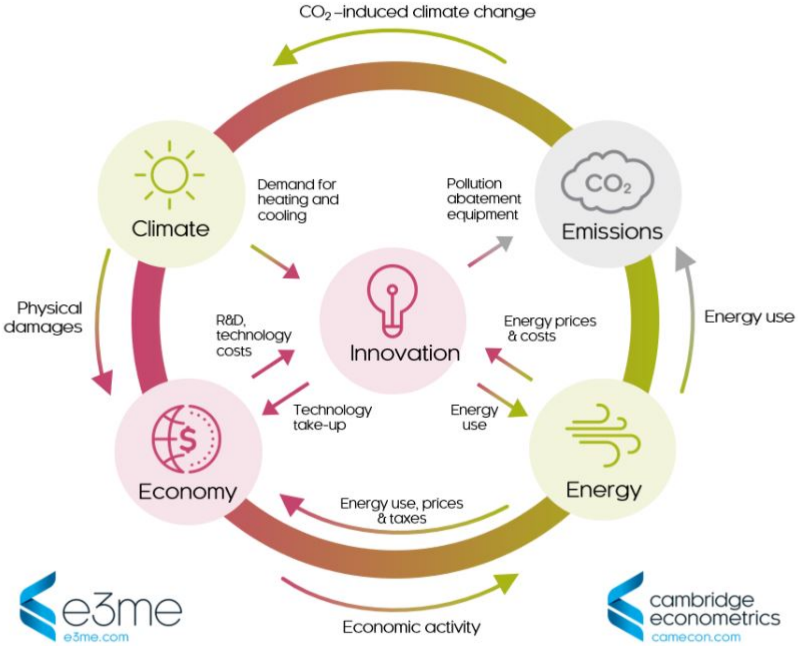

Figure 2.3: Cambrigde Econometrics e3me model, Source: Cambridge econometrics website

After this brief literature review, it can be seen that there is a gap on developing a model that provide an understanding of the Covid-19 economic impact. The most used model in epidemiology is the S.I.R. Model and after testing it it does not perform well when the basic reproduction number is above 3. After reading and checking Pollitt's work, the decision is clear: use a network-based model with differential equations that are non-linear and could describe the behaviour of the disease better. By modelling the disease the impact on the economy could be extrapolated demographics impact has a decisive impact on economies as [Okuyama and Sahin, 2009] and [Albala-Bertrand et al., 1993] said.

Chapter 3

Methodology

"Does a flap of a butterfly's wings in Brazil Set off a tornado in Texas?"

– Edward Lorenz, *1972*

3.1 Preliminary concepts

On this chapter I will start defining preliminary concepts that then will be applied on the model and the computation of the results. When deciding the methodology, I considered hints and future research lines from first class academics and also take into consideration the aforementioned literature.

3.1.1 Markov property and Markov chains

Let; $\{X(t), t = 0, 1, 2, ...\}$ be a Markov process if for any set of n time points $t_1 < t_2 < ... < t_n$ in the index set of the process, the conditional distribution of $X(t_n)$, for given values of $X(t_1), ..., Xt_{n-1}$, depends only on $X(t_{n-1})$, the most recent known value, for any real number $x_1, ..., x_n$.

$$P[X(t_n) \leq x_n | X(t_1) = x_1, ..., X(t_{n-1}) = x_{n-1}] \tag{3.1}$$

A real number x is said to be a possible value, or a state, of a stochastic process $\{X(t), t \in T\}$ if there exists a time t in T such that the probability $P[x - h < X(t) < x + h]$ is positive for every $h > 0$. The set of possible values of a stochastic process is called state space.

3.1.2 Law of large numbers and Central limit theorem

Bernoulli developed a theorem that stated: given the following arbitrary numbers $\epsilon > 0$ and $\eta > 0$, the following inequality holds.

$$P(|\frac{n_a}{n} - P_a| < \epsilon) \geq 1 - \eta \tag{3.2}$$

If:

$$n \geq \frac{1}{4\epsilon^2 \eta} \tag{3.3}$$

Being $P(A)$ The probability of a random event A and its frequency $\frac{n_a}{n}$ with large number of repeated experiments.

If the independent random variables $X_1, ..., X_n$ all have the same distribution with an expected value μ and Variance σ^2 then the distribution of the variable.

$$Y_n = \frac{\frac{1}{n}\sum_{i=1}^{n} X_i - \mu}{\sigma/\sqrt{n}} \tag{3.4}$$

Tends to the $(0, 1)$ normal distribution for $n \to \infty$.

3.1.3 First-order systems of ODE's

Let's consider an autonomous system of first-order ODE's of the form:

$$x_t = f(x) \tag{3.5}$$

Where $x(t) \in \mathbb{R}^d$ is a vector with dependent variables, $F : \mathbb{R}^d \to \mathbb{R}^d$ is a vector field, and dx/dt or $\dot{x}$ is the time-derivative in component from, $x = (x_1, ..., x_d)$,

$$f(x) = (f_1(x_1, ..., x_d), ..., f_n(x_1, ..., x_d)), \tag{3.6}$$

and the respective system is:

$$x_{1t} = (f_1(x_1, ..., x_d), \tag{3.7}$$
$$x_{2t} = (f_2(x_1, ..., x_d), \tag{3.8}$$
$$..., \tag{3.9}$$
$$x_{dt} = (f_d(x_1, ..., x_d), \tag{3.10}$$

Such that the equation (2.1) is describing the evolution in continuous time t of a dynamical system with finite dimensional state $x(t)$ of dimension d. This is an autonomous ODE and from them the models that arise are systems whose laws do not change in time, and they are invariant under translation in time. But a non autonomous system for $x(t) \in \mathbb{R}^d$ has the form:

$$x_{tt} = f(x, t) \tag{3.11}$$

where $F : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$. A. non-autonomous ODE describes systems governed by laws that vary in time. The equation (2.7) can be transformed into an autonomous sysystem for $y = (x, s) \in \mathbb{R}^n + 1$ with $s = t$ as:

$$x_t = f(x, s) \tag{3.12}$$
$$S_t = 1 \tag{3.13}$$

This increases the order of the system by one, and even if the original system has an equilibrium solution $x(t) = \overline{x}$ such that $f(\overline{x}, t) = 0$, the suspended system has no equilibrium solutions for y. Higher-order ODE's can be written as first order systems by the introduction of derivatives as new dependent variables.

3.1.4 Phase space and Flow maps

Very few nonlinear systems of ODE's are explicitly solvable. Therefore, rather than looking for individual analytical solutions, we try to understand the qualitative behavior of their solutions. This global, geometrical approach was introduced by Poincaré (1880).

Let $x(t; x0)$ denote the solution of the following initial value problem (IVP)

$$x_t = f(x)x(0) = x_0 \tag{3.14}$$

Defined on its maximal time-interval of existence $T_-(x_0) < t < T_+(x_0)$. The existence-uniqueness theorem implies that we can define a flow map or solution map,$\Phi_t : \mathbb{R}^d \to \mathbb{R}^d$ by:

$$\Phi_t(x_0) = x(t; x_0) \tag{3.15}$$

$$T_-(x_0) < t < T_+(x_0) \tag{3.16}$$

That is, Φ_t maps the initial data x_0 to the solution at time t. Note that $\Phi_t(x_0)$ is not defined for all $t \in \mathbb{R}, x_0 \in \mathbb{R}^d$ unless all solutions exist globally. The map depends on both the initial and final time, not just their difference.

3.2 Model

The model that best represents epidemics is the S.I.R. as long as the basic reproduction number is below 3, and that is not the case of Covid. According to [May and Oster, 1976][May, 2004], this models have a huge applications on the social sciences as well as modelling population growth. I also think that reshaping the S.I.R. Model into a dynamical system can represent the current outbreaks and the behaviour of pandemic on a more realistic way.

New approaches and modifications of the S.I.R. model have been appearing during the last years specially on this one. The models that Earn proposes [Earn et al., 2000] is the first step towards a model that can represent pandemics such as covid or much more complex infectious diseases.

The model that I propose consists of two integrated blocks the first block it is a Markov chain comparable to the S.I.R. model and the second part it is based on a non-linear dynamical system that allows the first part of the model to be representatives for basic reproduction rates above 1. Corona virus basic reproduction number is usually between 4 and 2.

3.2.1 Model using a Markov Chain

To model the spread of the virus I decided to use a kind of a stochastic process called Markov chain which is Markov process with discrete parameter and discrete state space.

Consider a discrete dynamical system which is observed at a discrete set of times. Let the successive observations be denoted by $X_0, X_1, ..., X_n$, X_n being a random system. The value of X_n represents the state at time n of the stochastic dynamical system. The sequence of $\{X_n\}$ is called a chain with only a finite countably infinite number of states in which the system can be. The following condition must be satisfied: for any integer $m > 2$ and any set of m points $n_1, n_2, ..., n_m$ the conditional distribution of X_{n_m}, for given values of $X_{n_1}, ..., X_{n_{m-1}}$, depends only on $X_{n_{m-1}}$.

$$P[X_m = x_m | X_0 = x_0, ..., X_{m-1} = x_{m-1}] \tag{3.17}$$

The deterministic model can be formulated by a system of differential equations, assuming that the population size is large and is differentiable in terms of time using the following equations:

$$\frac{d\phi}{dt} = -\theta i_t \phi I \tag{3.18}$$

$$\frac{dI}{dt} = \theta i_t \phi I - \omega_\delta I \tag{3.19}$$

$$\frac{d\delta}{dt} = \omega \delta I \tag{3.20}$$

$$\frac{d\alpha}{dt} = \omega \alpha I \tag{3.21}$$

Where:

- ϕ= Population not infected
- I= Infected
- δ= Dead
- α= Immunized
- θi_t=Transmission Rate
- $\omega\delta$= Mortality rate
- $\omega\alpha$= Immunization rate

The model is based on the following assumptions. Lets suppose that a number of infected persons is introduced into a community of individuals, susceptible to the disease in question. The disease spreads from the affected to the unaffected by contact infection. Each infected person runs through the course of his sickness, and finally is removed from the number of those who are sick, by immunity or by death. As the epidemic spreads, the number of unaffected members of the community becomes reduced. Since the course of an epidemic is short compared with the life of an individual, the population may be considered as remaining constant, except in as far as it is modified by deaths due to the epidemic disease itself. In the course of time the epidemic may come to an end. It is assumed that all members of the community are initially equally susceptible to the disease, and it will be further assumed that complete immunity is conferred by a single infection.

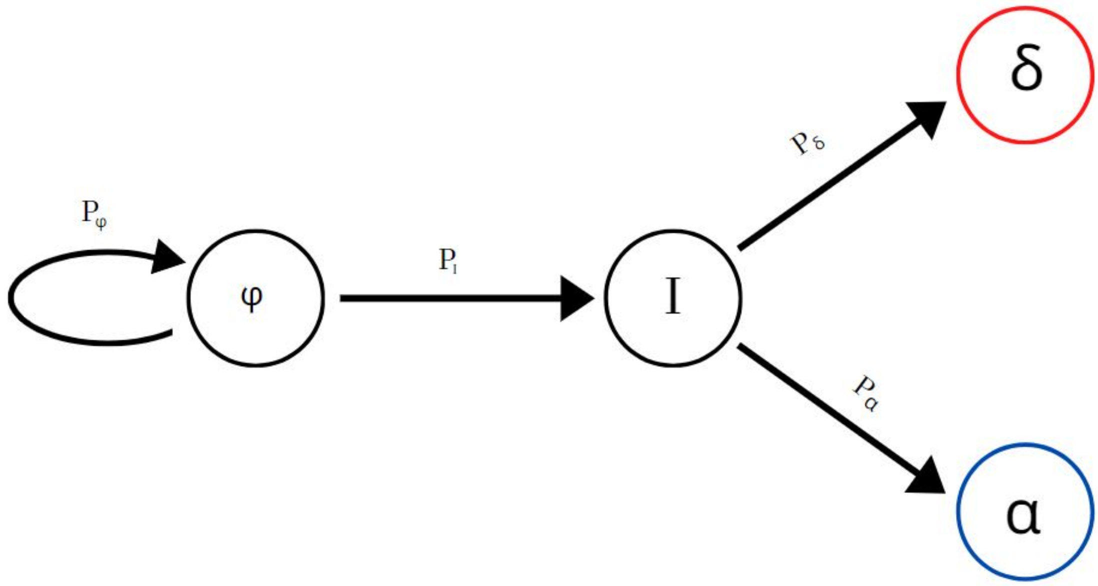

Figure 3.1: Markov Chain

Given the Law of large Numbers and Central Limit theorem, after successive trials, the reality tends to match the expected value. For example: lets suppose that you and me, we bet 1€ on tossing a coin the one who wins will receive 1€ from the one who lost. We assume that is a fair game so the $EV = 0$. On the short run it may happen that our balance differ from the expected value but on the long run our wallets tend to stay the same as the expected value which is equal that the one we had when we began the game. So in the case of the aforementioned model,

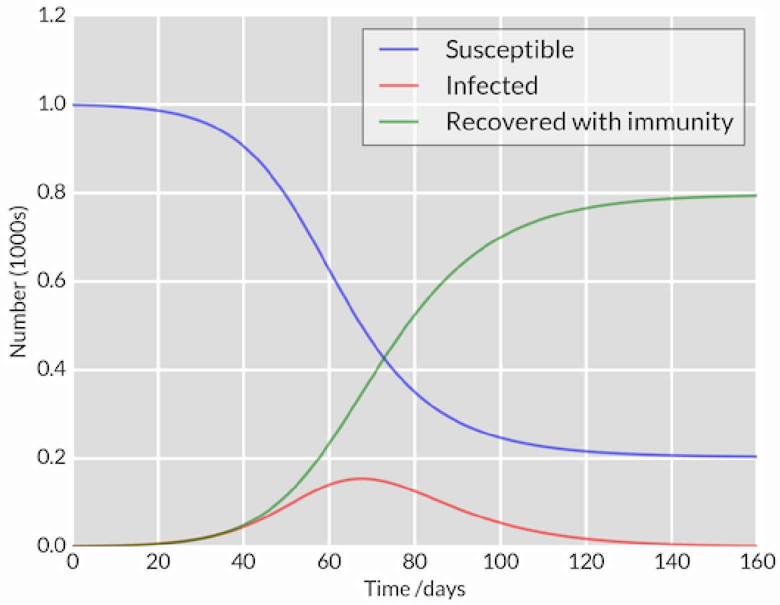

Figure 3.2: Behaviour of the Markov chain till equilibrium

On this case the probability of being infected is 0.8 and 0.2 the one of not being infected. At equilibrium reality match expected value.

3.2.2 Non-linear dynamical system

The non-linear dynamical system, it is composed by nonlinear equations of the form:

$$X_{t+1} = r \cdot X_t(1 - X_t) \tag{3.22}$$

Where:

- t_n, is the time while $n \in \{1, 2, ...\}$
- X_t is the quantity of a given state at time t while $X \in [0, 1]$
- The term r represents the reproduction rate at which the quantity X_t at period t grows or diminishes while $r \in [0, 4]$.

Equation (3.22) is widely used in several fields of applied mathematics. On social sciences and economics and also on epidemiology or modeling populations.

The first term ($X_{t+1} = r \cdot X_t$) behaves exponentially. But the key of this equation relies on introducing the second term $(1 - X_t)$ that we can impose a theoretical maximum for every X_t. When we analyse the behaviour, as the X_t grows bigger, it is harder to continue growing.

On the real life on the case of population modelling it states that resources are limited as Thomas Malthus said about population growth [Malthus, 1809]: " "I think I may make fairly two postulata. First, that food is necessary to the existence of man. Secondly, that the passion between the sexes is necessary and will remain nearly in its present state ... Assuming then my postulata as granted, I say, that the power of population is infinitely greater than the power in the earth to produce subsistence for man. Population, when unchecked, increases in a geometrical ratio. Subsistence increases only in an arithmetical ratio. A slight acquaintance with numbers will show the immensity of the first power in comparison of the second. By the law of our nature which makes food necessary to the life of man, the effects of these two unequal powers must be kept equal. This implies a strong and constantly operating check on population from the difficulty of subsistence. This difficulty must fall somewhere and must necessarily be severely felt by a large portion of mankind...". When modelling a population the second term of the equation is the key but when modelling the corona virus it is crucial for achieving greater curve fitting.

The following equation, is the one used for modelling the fraction of infected people of a population.

$$I_{t+1} = \theta i_t \cdot I_t \cdot (\Psi_t - I_t) \tag{3.23}$$

Where:

- t_n, is the time while $n \in \{1, 2, ...\}$.
- I_t is the fraction of infected people of a total population(Φ) at time t while $I_t \in [0, 1]$.
- The term θi_t represents the reproduction rate at which the quantity I_t at period t grows or diminishes while $\theta i_t \in [0, 4]$.

If what we attempt to model the fraction of a population of people susceptible of being infected we use the following equation:

$$\phi_{t+1} = 1 - (\theta i_t \cdot I_t \cdot (\Psi_t - I_t)) \tag{3.24}$$

Where:

- ϕ_t is the fraction of a population (Φ) of people susceptible of being infected at time t while $\phi \in [0, 1]$.
- t_n, is the time while $n \in \{1, 2, ...\}$.
- I_t is the fraction of infected people of a total population(Φ) at time t while $I_t \in [0, 1]$.
- The term θi_t represents the reproduction rate at which the quantity I_t at period t grows or diminishes while $\theta i_t \in [0, 4]$.

On the other side if we attempt to model is the fraction of a population that is immunized, we use the following equation:

$$\alpha_{t+1} = (\omega\alpha(\theta i_t \cdot I_t \cdot (\Psi_t - I_t)) \tag{3.25}$$

Where:

- α_t is the fraction of a population (Φ) that is immunized at time t while $\alpha \in [0, 1]$.
- t_n, is the time while $n \in \{1, 2, ...\}$.
- I_t is the fraction of infected people of a total population(Φ) at time t while $I \in [0, 1]$.

- The term θi_t represents the reproduction rate at which the quantity I_t at period t grows or diminishes while $\theta i_t \in [0, 4]$.

- $\omega\alpha$ represents the immunization rate of the disease

On the other side if we attempt to model is the fraction of a population that is immunized, we use the following equation:

$$\delta_{t+1} = (\omega\delta(\theta i_t \cdot I_t \cdot (\Psi_t - I_t)) \tag{3.26}$$

Where:

- δ_t is the fraction of a population (Φ) that passes away at time t while $\delta \in [0, 1]$.
- t_n, is the time while $n \in \{1, 2, ...\}$.
- I_t is the fraction of infected people of a total population(Φ) at time t while $I \in [0, 1]$.
- The term θi_t represents the reproduction rate at which the quantity I_t at period t grows or diminishes while $\theta i_t \in [0, 4]$.
- $\omega\delta$ represents the mortality rate of the disease

According to the population model the term Ψ stands for the theoretical maximum which is equal to 1 for every period of time t_n.

$$\Psi_t = \phi_t - \alpha_t - \delta_t = 1 \tag{3.27}$$

Other variables that appear on the model are defined on the following list:

- let θi_t be the basic reproduction number of the virus(R) and $\theta i_t \in [0, 4]$.

$$\theta i_t = \frac{I_{t+1}}{I_t} \tag{3.28}$$

- let $\omega\delta$ be the death rate of the virus and $\phi_{t+1} \in [0, 1]$.

$$\omega\delta = \frac{\delta}{I} \tag{3.29}$$

- let $\omega\alpha$ be the immunization rate and $\phi_{t+1} \in [0, 1]$.

$$\omega\alpha = \frac{\alpha}{I} \tag{3.30}$$

A markov property is also introduced into this model:

$$P[X_m = x_m | X_0 = x_0, ..., X_{m-1} = x_{m-1}] \tag{3.31}$$

For every time period t the nodes of t will be depending only of the node of the previous time period $t-1$. With only three possible results for every period depending on the dynamic and corresponding probabilities associated to each state:

- State I_t with probability P_{It} where:

$$P_{It} = \frac{I_t}{I_t + \phi_t} \in [0, 1] \tag{3.32}$$

- State ϕ_t with probability $P_{\phi t}$ where:

$$P_{\phi t} = 1 - (\frac{I_t}{I_t + \phi_t}) \in [0, 1] \tag{3.33}$$

- State δ_t with probability $P_{\delta t}$ where:

$$P_{\delta t} = (\frac{I_t}{I_t + \phi_t}) \cdot (\frac{\delta_t}{\delta_t + \alpha_t}) \in [0, 1] \tag{3.34}$$

- State α_t with probability $P_{\alpha t}$ where:

$$P_{\alpha t} = (\frac{I_t}{I_t + \phi_t}) \cdot (\frac{\alpha_t}{\delta_t + \alpha_t}) \in [0, 1] \tag{3.35}$$

3.3 Data collection

The data used for testing the model it was extracted from the website: https://ourworldindata.org/. Data-set was all composed by the official information provided by 207 countries daily from the first case detection till the 1st of may. The site, it was deeply recommended to me by Marta Trapero a former lecturer at Universitat International de Catalunya and an experienced researcher. She recommended me the website as a source of data due to the certainty. I contrasted the our world in data data-set with the Johns Hopkins coronavirus resource centre but I decided to use the our world in data. The underlying reason was because of the accuracy and also because the days where data was missing from third world countries, our world in data they all-ready estimated a value. Google as a source also uses the our world in data and that also made me trust more the aforementioned source. The data regarding economic indicators: GDP, HDI,... was also extracted from the same site.

3.4 Data analysis

All the data analysis was done using Python 3.0. Due to its simpleness and the huge variety of modules for data analytics. On the internet and some editorials post resources online, so it was the best option for me. Considering that I all-ready knew some basic functions.

The modules used where the following:

- Panda to do manage all the dataset and do the analytics of it.
- The module of numpy to test the model and do numerical calculations.
- All the plots have been done using the matplotlib module.

All the coding has been done on the Jupiter notebooks from anaconda navigator. The reason for this it is because working on a cloud it is really comfortable for me and specially when the data-set is such heavy in terns of the size of the file.

Chapter 4

Results

"If you want different results, do not do the same things."

– Albert Einstein

On this chapter the results will be commented and analysed. The results where obtained after defining the model in python programming language. By using Jupiter notebooks, and some python modules the outcome was obtained as well as the pictures of the plots on this chapter.

As mentioned on the preliminary concepts of the previous chapter most differential equations, do not have an explicit solution. Phase space and flow maps or solution maps have been used in order to draw an analytical solution to the model. By thus the qualitative behaviour of the solution can be understood.
The findings of this dissertation have been grouped into sections each for each kind of outcome obtained.

4.1 Shape of the equilibrium

The equilibrium of every state X given r and X_0 is defined by the equation (4.1):

$$X_{t+1} = r \cdot X_t(1 - X_t) \tag{4.1}$$

Equation (4.1) is widely used in several fields of applied mathematics. On social sciences and economics and also on epidemiology or modeling populations.

The first term ($X_{t+1} = r \cdot X_t$) behaves exponentially and it is by introducing the second term ($1 - X_t$) that we can impose a theoretical maximum for every X_t. As time passes reality will converge towards the equilibrium X_e as $t \to \infty$.

On Figure 4.1, the vertical axis represents the frequency and on the horizontal axis the equilibrium. This plot represents how complex are the dynamics of the model even considering the simpleness.

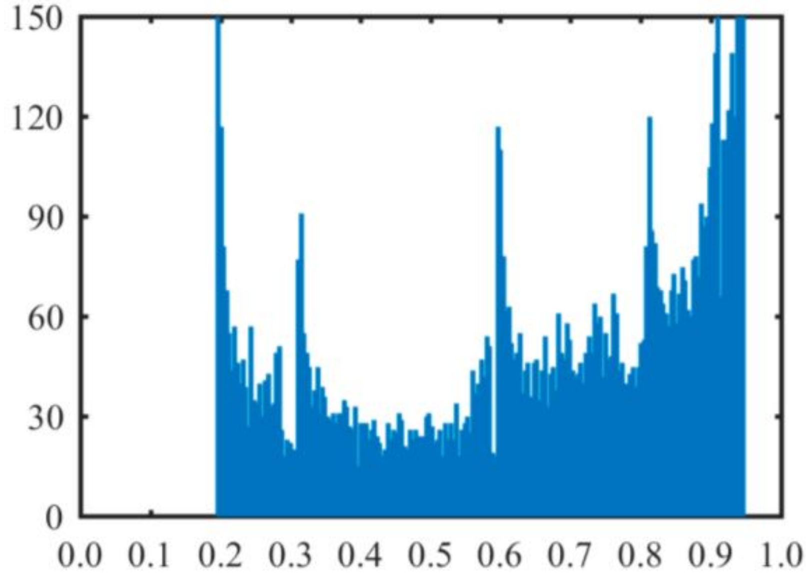

Figure 4.1: Frequency distribution of equilibrium

4.2 Path towards equilibrium X_e

After seeing figure 4.1, complexity's and the lack of clearness.In order to understand better how it is the equilibrium of the model, it will be proceeded to plot a cobweb plot to see and understand the qualitative behaviour of equation (6.1) towards its equilibrium.

On the papers of dynamical systems, the cobweb plot it is widely used among academics. This plot helps to understand given different initial conditions of a non-linear dynamical system how it reaches the equilibrium point.

To plot the following figures, It is considered the initial population it is $X_0 = 0.08$, which is comparable to the current situation now at Spain.

For r=0

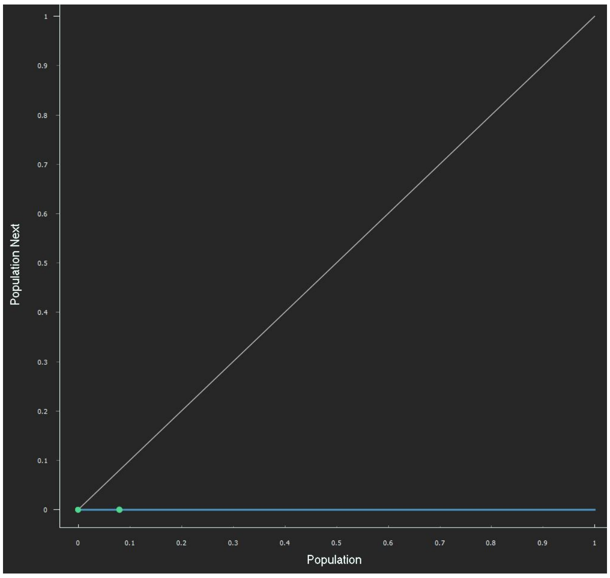

Figure 4.2: Cobweb plot for r=0

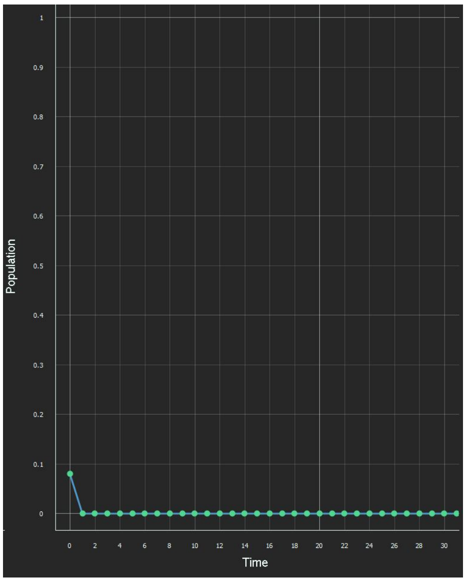

Figure 4.3: X vs Time for r=0

With r being 0, the population will die, independent of the initial population.

If r = [0,1]

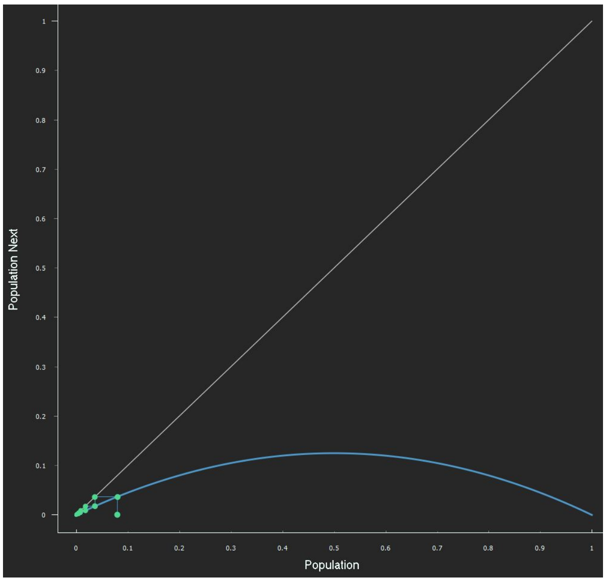

Figure 4.4: Cobweb plot for r=0.5

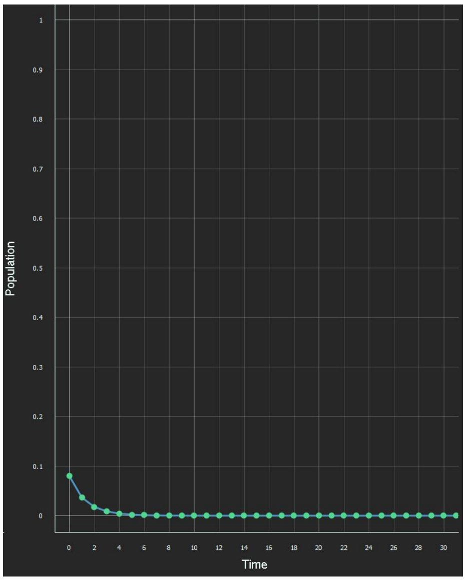

Figure 4.5: X vs Time for r=0.5

With r between 0 and 1, the population will eventually die, independent of the initial population.

If r= (1,2]

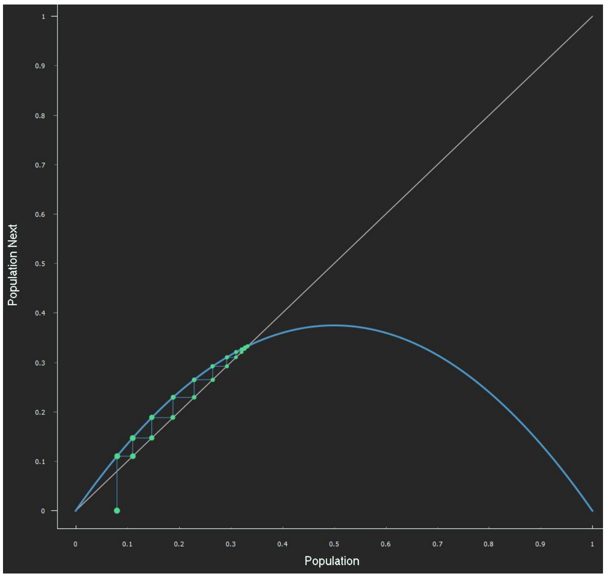

Figure 4.6: Cobweb plot for r=1.5

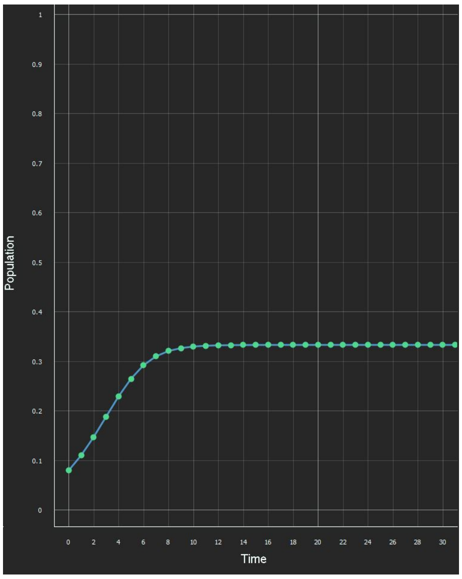

Figure 4.7: X vs Time for r=1.5

If $r \in (1, 2]$; $X_e = \frac{r-1}{r}$. The equilibrium population tends $\frac{r-1}{r}$, regardless of the value initial population X_0.

If r= (2,3]

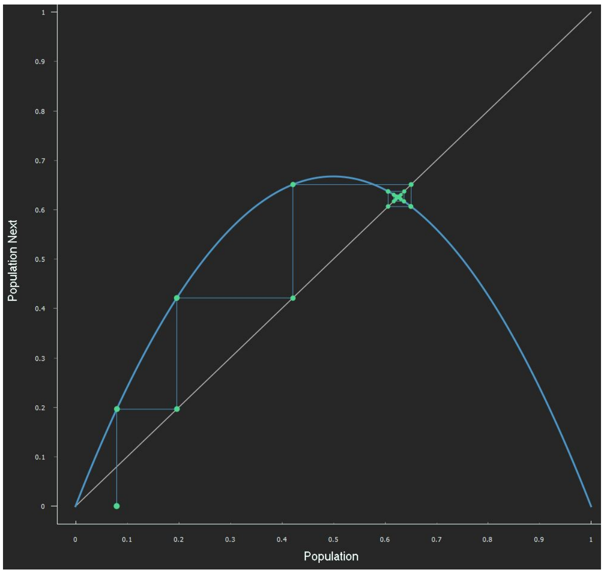

Figure 4.8: Cobweb plot for r=2.5

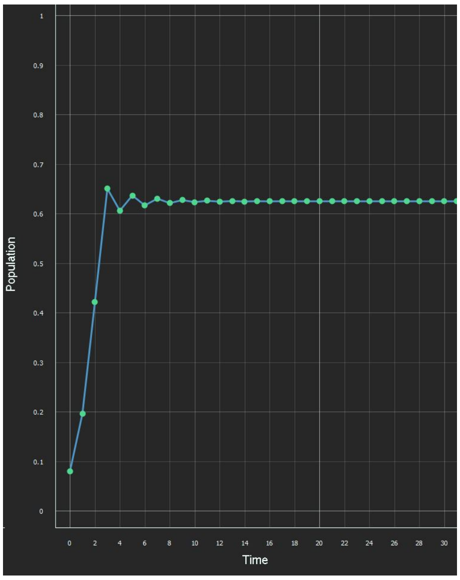

Figure 4.9: X vs Time for r=2.5

If $r \in (2,3]$; $X_e = \frac{r-1}{r}$. The equilibrium population tends $\frac{r-1}{r}$, regardless of the value initial population X_0 but will be fluctuating around for some period of time t.

If r= (3,3.45]

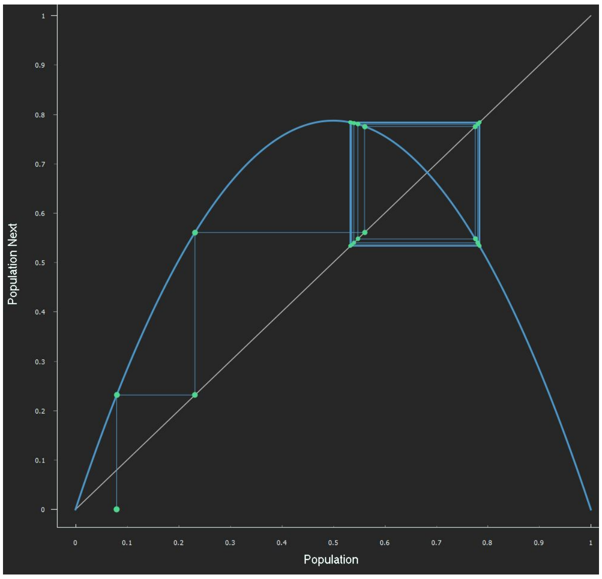

Figure 4.10: Cobweb plot for r=3.15

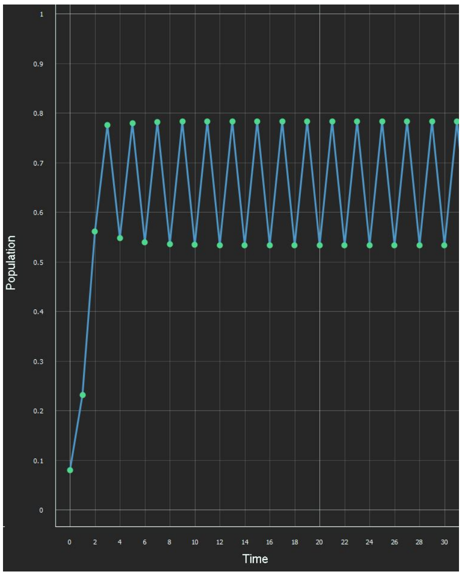

Figure 4.11: X vs Time for r=3.15

If $r \in (3, 1+\sqrt{6}]$; The equilibrium will be permanently oscillating among two values that depend on r.

If r= (3.45,3.54]

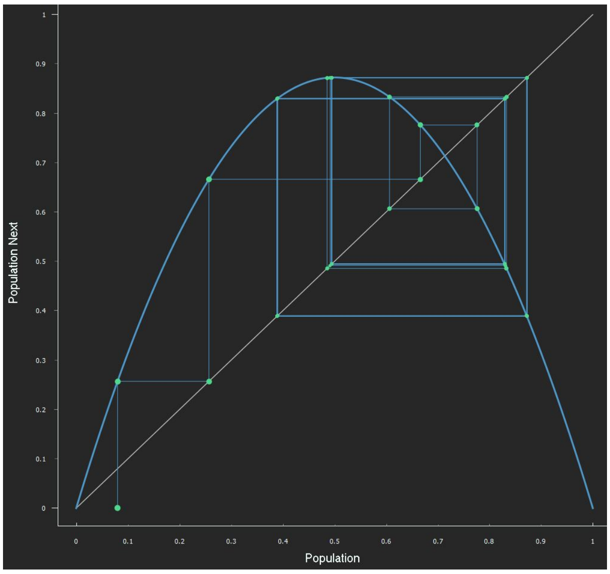

Figure 4.12: Cobweb plot for r=3.5

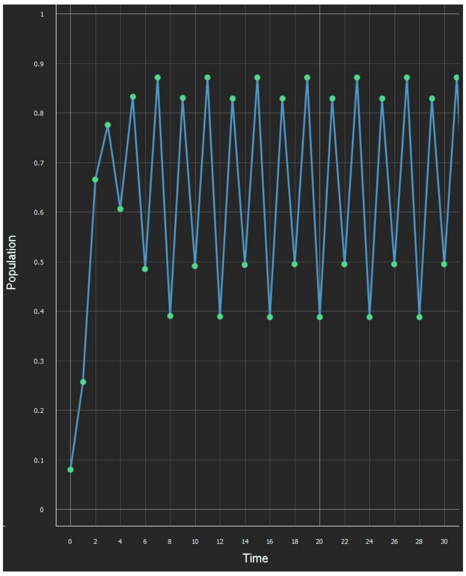

Figure 4.13: X vs Time for r=3.5

If $r = 3.5$; The equilibrium will be oscillating among 4 values.

If r= 3,55

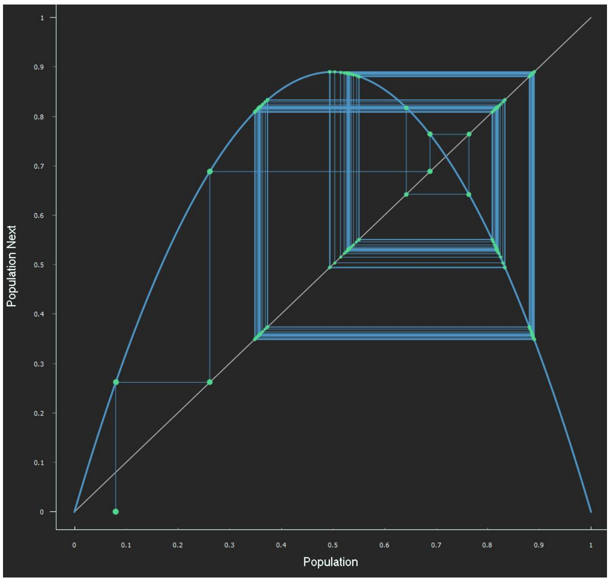

Figure 4.14: Cobweb plot for r=3.55

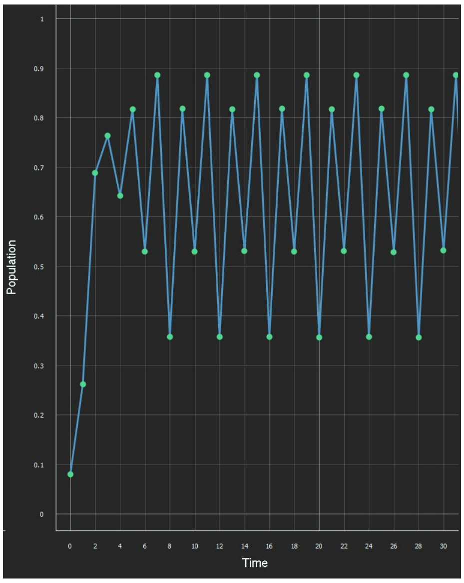

Figure 4.15: X vs Time for r=3.55

If $r \in (3.54409, 3.56995]$; The equilibrium will be oscillating among 8, then 16, 32,...

If r=3.57

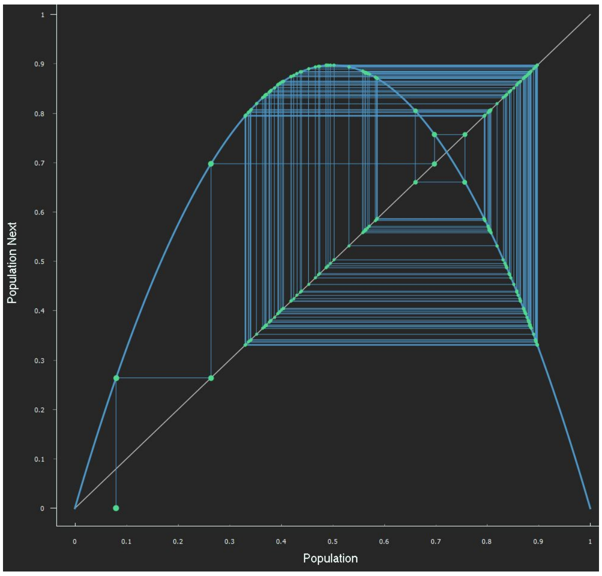

Figure 4.16: Cobweb plot for r=3.57

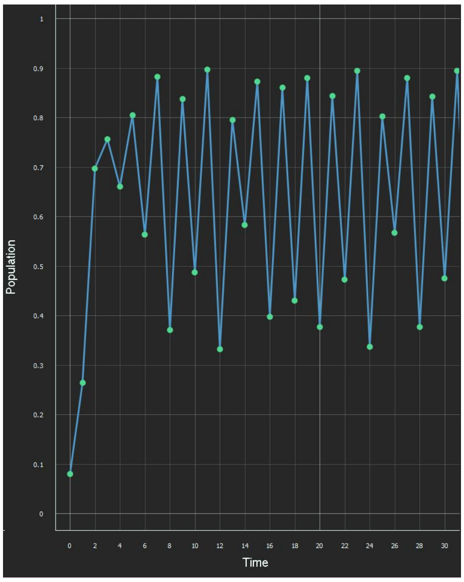

Figure 4.17: X vs Time for r=3.57

If $r > 3.56995$; It adopts chaotic behaviour, a slight variation on the initial conditions would yield into a dramatically different result

If r=4

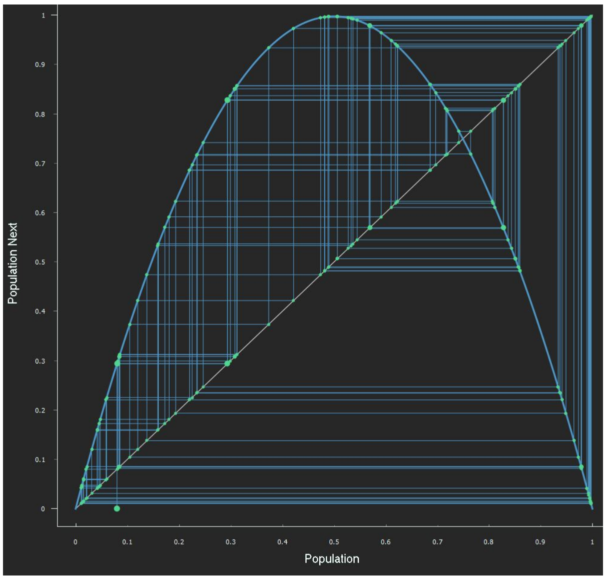

Figure 4.18: Cobweb plot for r=4

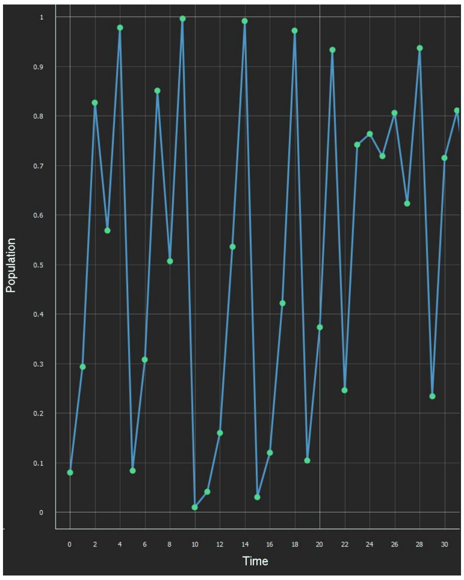

Figure 4.19: X vs Time for r=4

If $r > 3.56995$; It adopts chaotic behaviour, a slight variation on the initial conditions would yield into a dramatically different result

4.3 Logistic map

If we plot all the equilibrium $X_e \in [0, 1]$ for every $r \in [0, 4]$ given a initial population $X_0 \in [0, 1]$ it will result what is called the logistic map. if we take the logistic map differential equation and we solve-it; we will get the logistic map equation which is the same equation used on the model.

$$\frac{d}{dx} f(x) = f(x)(1 - f(x)) \tag{4.2}$$

With a boundary condition: $f(0) = 1/2$

$$X_{t+1} = r \cdot X_t(1 - X_t) \tag{4.3}$$

When $t \to \infty$

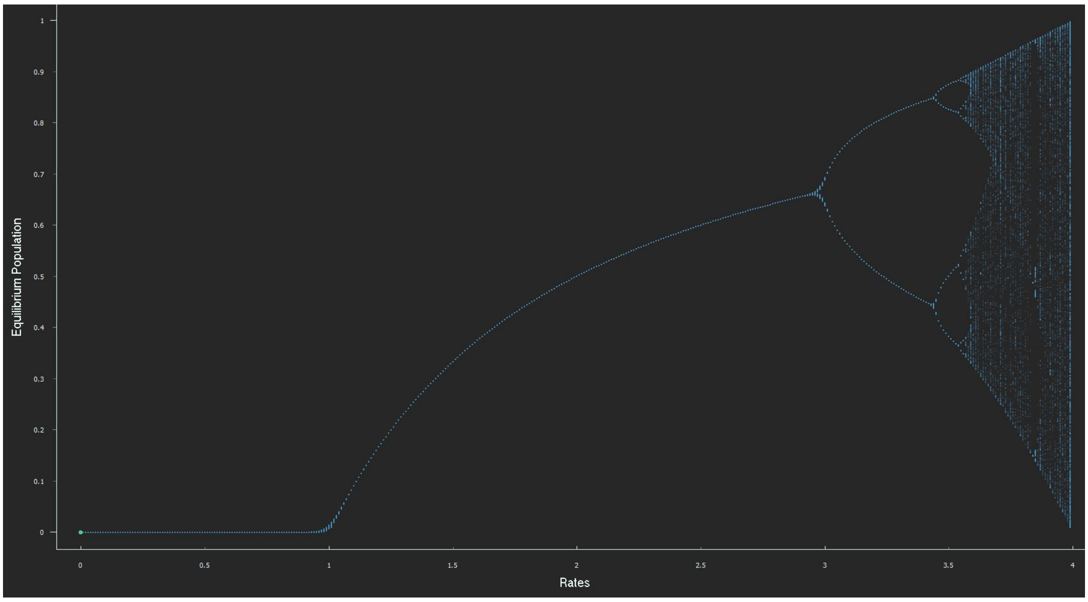

Figure 4.20: Equilibrium population vs r

- If $r \in [0, 1]$; $X_e = 0$. The equilibrium population tends to 0.
- If $r \in (1, 2]$; $X_e = \frac{r-1}{r}$. The equilibrium population tends $\frac{r-1}{r}$, regardless of the value initial population X_0.
- If $r \in (2, 3]$; $X_e = \frac{r-1}{r}$. The equilibrium population tends $\frac{r-1}{r}$, regardless of the value initial population X_0 but will be fluctuating around for some period of time t.
- If $r \in (3, 1 + \sqrt{6}]$; The equilibrium will be permanently oscillating among two values that depend on r.
- If $r \in (1 + \sqrt{6}, 3.54409]$; The equilibrium will be permanently oscillating among four values.
- If $r \in (3.54409, 3.56995]$; The equilibrium will be oscillating among 8, then 16, 32,...
- If $r > 3.56995$; It adopts chaotic behaviour, a slight variation on the initial conditions would yield into a dramatically different result.

After researching it seems that the logistic differential equation it is widely used on several fields of the knowledge and has multiple applications. I will just note some applications of it and fields that is used:

- In ecology for modelling population growth and for time-varying carrying capacity.
- Statistics and machine learning used to model the chance a chess player has to beat his opponent.
- In chemistry: reaction models.
- Neural networks to introduce non linearity in the model.
- In medicine: modeling of growth of tumors.
- In economics and sociology: diffusion of innovations.
- In linguistics: language change.
- In agriculture: modeling crop response.

4.4 Feigenbaum constant and Fractals

During 1975 Mitchell J. Feigenbaum, discovered what today is called the feigenbaum constant, but was on 1978 [Feigenbaum, 1978] that he published the discovery. The Constant arise from the logistic map. The ratio by which the period-doubling bifurcations is from where the constant arises. It is surprising that inside a chaotic dynamical system there is a constant.

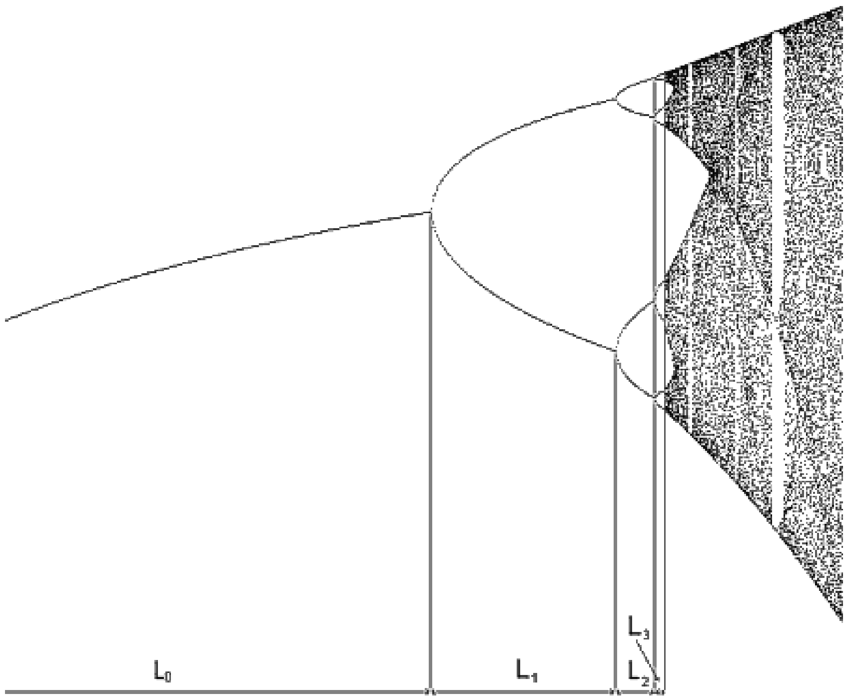

Figure 4.21: Feigenbaum constant on the logistic map

After the discovery of the aforementioned constant the only way to have an infinite function on a limited place is by a mathematical structure which is a fractal. Science then researchers where able to connect the logistic map with one of the most famous fractals which is the Mandelbrot set [Mandelbrot, 2013].

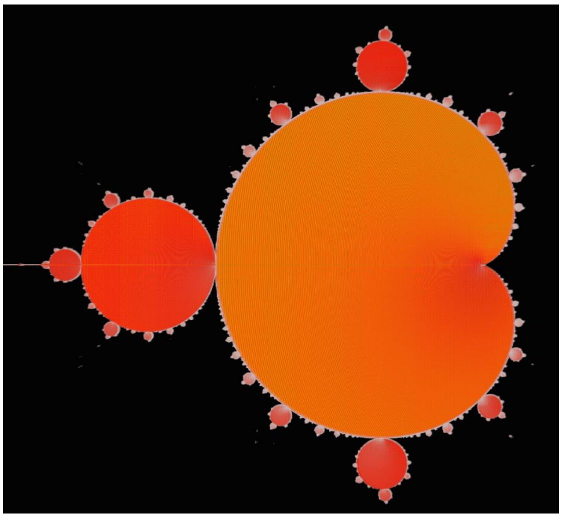

Figure 4.22: Mandelbrot set

Figure 4.23: Logistic map and Mandelbrot set

4.5 Correlation matrix of the data-set

The data-set used was based on officially daily reported data of 207 countries. The data-set was complimented from economic indicators, vaccination data till 1st of May of 2021.
The following figure is the correlation matrix of more that 5 million data-points.

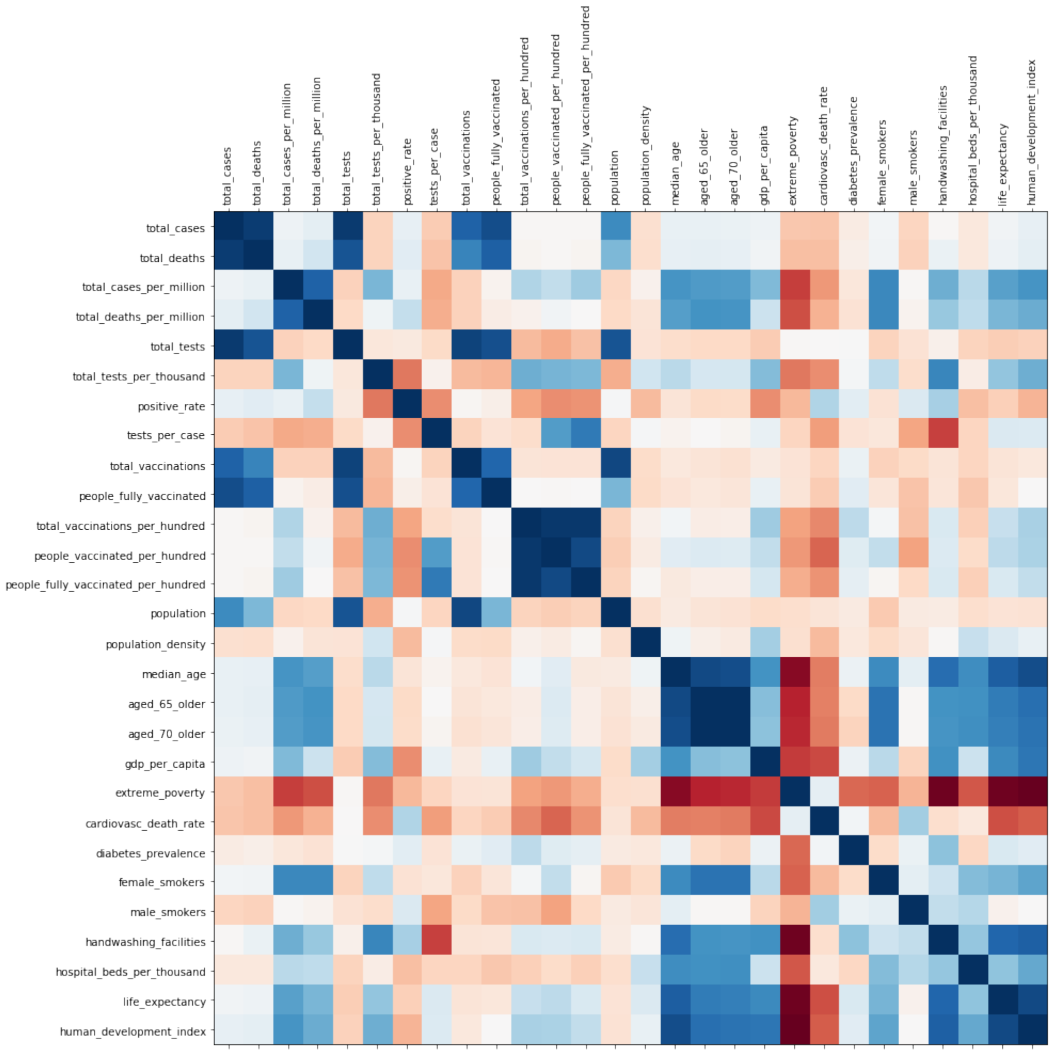

Figure 4.24: Correlation matrix of the data-set

The main conclusion from the Figure 4.24 is that the higher the HDI the higher the cases as well as deaths per million inhabitants. richer countries are going through the pandemic really fast, but they are proving to be able to fight it really hard too. In my opinion developing countries are generally going slower through the pandemic and it is mode difficult for them to fight against it. Developing countries must be helped because the mortality rate it is way higher.

Chapter 5

Discussion and conclusions

> *"It is not the strongest of the spices the one that survives, nor the most intelligent that survives. It is the one that is most adaptable to change."*
>
> – Charles Darwin

It amazes me how a relative simple model can exhibit that complex dynamics, but reality is much more complex than this model. What leaves to us clear this model that I share with the Cambridge econometric is that in some cases and specially in those where there is such a complexity and volatility that the best advise you cloud give to policymakers is to stay prepared for any scenario. Tiny differences on the initial conditions could yield to non comparable outcomes on the long run and specially if the reproduction rate is above certain values.

On the model built, has a strong connection with the S.I.R. model regarding the approach on defining the situation both based on a markov chain. To incorporate dynamical systems and present the model as a system of differential equations allows the model to go a step further and to behave on a more realistic way, with a behaviour similar to some models proposed by Lorenz as well as the logistic map and it's period doubling constant. The impact of the virus is determined by the epidemic itself and the differential equations have a strong connection with other models proposed on the modelling of trends such as the ice-bucket challenge and similar situations where the behaviour follows a pattern with three conditions: one, contagiousness; two, the fact that little causes can have big effects; and three, that change happens not gradually but at one dramatic moment.

The economic impact is almost fully determined by the demographic impact as well as the ability for the country to overcome unexpected situations. As the system becomes volatile, the complexity increases as well as the uncertainty and expectations and this has a determinant effect on the way economics work. I can say that under small basic reproduction number values, the situation is under control in the short run but as what the model shows us is that things can change from the night to the day.

This exciting and challenging project has kept me learning from the very first day until its end and all the knowledge that I acquired during this path is of great value for my professional future. Trying to define a complex problem that implicates almost any human on the earth and do your best to find a solution to a real problem has made me grow.

My recommendation to economic agents based on the research is that they must be able to quickly adapt and adjust to the new market conditions because the equilibrium is constantly changing and it can become really volatile. Policymakers must be able to mitigate the volatility by policy-making and influencing the expectations and uncertainty but as the model showed to us the most important thing is to be able to keep the situation under control.

Reality is really complex and future is uncertain specially in some situations. I am very proud of myself to see that the model that I build represents the unpredictable reality on the long run. I think that the value that brings this model to the table is amazing. It helps decision makers to clearly understand the behaviour of the situation and to see the reality that will come. In some situations this information is invaluable specially if it is about saving lives.

5.1 Future research and limitations

The main limitation of this project was time. I has been hard to me to be finishing the engineering program in Italy and at the same time to be researching and try to develop this model.

The data available it is not scarce in terms of volume put the main point is that it is mainly data on how the virus spreads but not how we fight the virus because we are still on an early stage.

The subjects involved on this project and the research require the ability to read and understand mathematics papers which in my case at the beginning it wasn't easy.

For future research as the data becomes available on how the countries recover the model could be expanded and also finally completed to measure the impact of a pandemic because the only way that I consider it can be measured at this point is to measure the impact of the virus by attempting to model itself.

In the field of stochastic processes, I consider that it also could be used A continuous time stochastic process such as brownian motion and monte-carlo simulation for a model with continuous-time and with more complexities. As a limitation it is needed expertise on continuous-time stochastic process, probability theory and high modelling capabilities.

I consider that developing an algorithm with neural networks and artificial intelligence connected to a large data-set regarding several policies applied at specific regions, will be really useful for policy-makers in order to mitigate the impact and to optimize response with reinforced learning. Specially in complex situations like this.

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